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ERROR BOUNDS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS

by

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A THESIS

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The undersigned certify that they have read,
recommend to the Faculty of Graduate Studies for acceptance,
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of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

A system of n first order linear differential equations in matrix form

$$(*) \quad \frac{dW}{dz} = z^h \left[\sum_{k=0}^{s-1} A_k z^{-k} + A_s(z) z^{-s} \right] W \quad (s = 0, 1, 2, \dots)$$

where h is an integer, $A_0(z) = A(z)$, each A_k ($k = 0, 1, 2, \dots$) is a constant $n \times n$ matrix, the n^2 elements of $A_s(z)$ ($s = 0, 1, 2, \dots$) are holomorphic in some domain $\mathcal{D}$ extending to infinity and for z in $\mathcal{D}$, $\lim_{z \rightarrow \infty} |\{A_s(z)\}_{ij}| = \kappa_{ijs} < \infty$, has formal solutions which are asymptotic expansions (as $|z| \rightarrow \infty$) of actual solutions in certain sectors extending to infinity in $\mathcal{D}$.

In this thesis known results of error bounds for partial sum approximations of formal solutions of second order equations to actual solutions of second order equations are extended to the general system (*). This is achieved by first transforming the system (*) to canonical form. Formal solutions are then explicitly constructed for the transformed system (henceforth to be referred to as (**)) and error bounds are obtained for partial sum approximations of these formal solutions to actual solutions of (**).

The treatment of the general case (*) is preceded by the treatment of the special cases $h = -2$ (which includes $h \leq -2$), $h = -1$, and arbitrary non-negative h with the eigenvalues of A_0 distinct. The case $h = -2$ is ready for formal solution as it stands. Transformations are explicitly constructed which transform the particular $h \geq 0$ case when the

eigenvalues of A_0 are distinct and the $h = -1$ case to canonical form. A known existence theorem concerning transformation of the general $h \geq 0$ case is stated and a modification of this theorem corresponding to our modification of the usual $h = -1$ canonical form is proved.

Assuming that we require actual solutions of (**) for which the formal solutions are asymptotic expansions (as $|z| \rightarrow \infty$), and assuming the domain $\mathcal{D}$ is sufficiently large it is always possible in the special cases $h = -2$ and $h = -1$ to bound the error vector by use of a single Volterra vector integral equation. A simple geometric criterion is given by which one can determine whether a single, or a simultaneous pair of Volterra vector integral equations are required to express the error in the case when $h \geq 0$ and the eigenvalues of A_0 are distinct. In all these special cases it is possible to choose the end-points of integration at infinity. This is no longer always possible in the general case, even if $\mathcal{D}$ is a neighborhood of infinity. Although it may be necessary to choose one finite end-point of integration it is nevertheless always possible to express the error vector by at most a simultaneous pair of Volterra vector integral equations.

As an example, the general $n = 2$ case is completely solved, including a flow chart illustration of the transformation of this case to canonical form.

Real variable inequalities useful for bounding solutions of integral equations are obtained.

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CHAPTER I

INTRODUCTION

This introductory chapter is intended to facilitate the understanding of the thesis.

Some definitions and notations are established.

A brief review of the literature leading to the problem of this thesis is given.

The problem, namely that of obtaining a bound on the difference between actual solutions and partial sums of formal solutions is described, and the need for solution of this problem is emphasized.

A preview of the organization of the remainder of the thesis is given.

1.1 Matrix Notations and Norms

We shall use the following notations to refer to an $m \times n$ matrix A :

$$(1.1.1) \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

To refer to the (i,j) 'th element of a matrix A , we shall use

$$(1.1.2) \quad \{A\}_{ij} = a_{ij}.$$

If the matrix A is square ($m = n$) we designate the unit matrix consisting of ones along the diagonal and zeros elsewhere by I .

The diagonal part of the square matrix A with diagonal elements

$a_{11}, a_{22}, \cdots, a_{nn}$ is designated by $[a_{11}, a_{22}, \cdots, a_{nn}]$, or simply

by D when it is convenient not to illustrate the diagonal elements.

Furthermore by $\bar{A}$ we shall mean $A = [a_{11}, a_{22}, \dots, a_{nn}]$ where $A = [a_{ij}]$.

We shall also use the notation

$$(1.1.3) \quad A = [A_{ij}] ; \quad i, j = 1, 2, \dots, \ell$$

where A_{ij} is a $\mu_i \times \mu_j$ matrix for some positive integers μ_i and μ_j , $\mu_1 + \mu_2 + \dots + \mu_\ell = n$.

In carrying out error analysis we shall often replace an $m \times n$ matrix A by its "comparison matrix" $|A|$ where

$$(1.1.4) \quad |A| = [|\{A\}_{ij}|]$$

i.e. $|A|$ is the matrix of absolute values of the elements of A . This applies equally to the case when m is an arbitrary positive integer and n is 1, i.e. to the case when A is a vector.

Given two $m \times n$ real matrices A and B of the same dimension, we interpret $A \leq B$ to mean $\{A\}_{ij} \leq \{B\}_{ij}$ for each (i, j) ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$). We note that this definition includes the possibility that A and B are vectors.

It will often be convenient to use norms of vectors and matrices. We assume $A = [a_{ij}]$ be an $m \times n$ matrix. Possible matrix norms $\|A\|$ for A are:

$$\begin{aligned}
 (a) \quad \ell_1 \text{ norm} \quad \|A\| &= \max_j \sum_{i=1}^m |a_{ij}| \\
 (1.1.5) \quad (b) \quad \ell_\infty \text{ norm} \quad \|A\| &= \max_i \sum_{j=1}^n |a_{ij}| \\
 (c) \quad \ell_2 \text{ norm} \quad \|A\| &= \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}
 \end{aligned}$$

We again note that the definitions (1.1.5) include the case of A being a vector.

Each definition of $\|A\|$ in (1.1.5) satisfies the following properties:

$$\begin{aligned}
 (a) \quad \|A\| &\geq 0 \quad \text{and} \quad \|A\| = 0 \quad \text{if and only if} \quad A = 0 \\
 (b) \quad \|cA\| &= |c| \|A\|, \quad c \text{ is a complex number} \\
 (1.1.6) \quad (c) \quad \|A + B\| &\leq \|A\| + \|B\|, \quad A \text{ and } B \text{ both } m \times n \\
 (d) \quad \|AB\| &\leq \|A\| \|B\|, \quad A \text{ is } m \times k, \quad B \text{ is } k \times n.
 \end{aligned}$$

The definitions (a) and (b) of $\|A\|$ in (1.1.5) are readily shown to satisfy (1.1.6). Case (c) of (1.1.5) clearly also satisfies (a), (b) and (c) of (1.1.6). With $B = [b_{ij}]$, the proof for (d) of (1.1.6) is given in [12], page 228.

Combining the definitions of (1.1.4) and (1.1.5), it follows that (1.1.7) $\| |A| \| = \|A\|$.

1.2 Some Transformations on a 2×2 Matrix

A general theory of functions of matrices is presented in the first volume of [5]. In this section we shall consider some functions of the matrix

$$(1.2.1) \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where α , β , γ and δ are non-negative numbers. Suppose this matrix has distinct eigenvalues. Then there exists a non-singular matrix T

$$(1.2.2) \quad T = \begin{pmatrix} 1 & u \\ v & 1 \end{pmatrix}; \quad T^{-1} = \frac{1}{1 - uv} \begin{pmatrix} 1 & -u \\ -v & 1 \end{pmatrix}$$

such that

$$(1.2.3) \quad T A T^{-1} = \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$$

where Λ_1 and Λ_2 are the eigenvalues of A .

Substituting (1.2.2) into (1.2.3) and equating off-diagonal elements in the resulting matrix to zero, we find that a solution for T is obtained if we choose u and v as follows:

$$(1.2.4) \quad \begin{aligned} u &= \frac{1}{2\gamma} \{ \alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} \\ v &= -\frac{1}{2\beta} \{ \alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} . \end{aligned}$$

We find moreover by back substitution that

$$(1.2.5) \quad \begin{aligned} \Lambda_1 &= \frac{1}{2} \{ \alpha + \delta - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} \\ \Lambda_2 &= \frac{1}{2} \{ \alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} . \end{aligned}$$

Let $f(A)$ be defined by

$$(1.2.6) \quad f(A) = \sum_{k=0}^{\infty} c_k A^k ,$$

where $\sum_{k=0}^{\infty} |c_k| \Lambda^k$ is convergent for $|\Lambda| < \Lambda_2$. Then using

$f(A) = T f(TAT^{-1})T$ or more directly using the Lagrange-Sylvester interpolation formula (see e.g. [15] p. 77) we find that

$$(1.2.7) \quad f(A) = \frac{(A - \Lambda_1) f(\Lambda_2) - (A - \Lambda_2) f(\Lambda_1)}{\Lambda_2 - \Lambda_1} .$$

If the eigenvalues of A are not distinct, an arbitrarily small change in the elements of A can make them distinct. Hence by continuity of (1.2.7) equation (1.2.7) holds whether the eigenvalues of A are distinct or not. For if they are not distinct we can treat (1.2.7) as if they were and write this equation in the form

$$(1.2.8) \quad f(A) = (A - \Lambda_1) \left\{ \frac{f(\Lambda_1) - f(\Lambda_2)}{\Lambda_1 - \Lambda_2} \right\} - f(\Lambda_2) .$$

Upon noting that $f(\Lambda)$ is an analytic function of the complex variable Λ we obtain, taking the limit $\Lambda_2 \rightarrow \Lambda_1$,

$$(1.2.9) \quad f(A) = (A - \Lambda_1 I) f'(\Lambda_1) - I f(\Lambda_1) .$$

1.3 Variations on a Contour

Let $f(z)$ be an analytic function of z and let $\mathcal{P}$ be a contour such that the integral on the right of (1.3.1) exists. We define

$$(1.3.1) \quad \gamma_{\mathcal{P}}(f(z)) = \int_{\mathcal{P}} \left| \frac{d}{dz} f(z) dz \right| = \int_{\mathcal{P}} |d f(z)|$$

where the contour $\mathcal{P}$ lies in the domain of analyticity of $f(z)$.

1.4 Formal Series and Asymptotic Expansions

We shall use the definition of formal series given in [4] (pages (115) and (141)).

Poincaré's [38] definition of an asymptotic expansion will be employed. More general definitions are given in [25,28]. A formal series

$$(1.4.1) \quad \tilde{F}(z) \hat{=} \sum_{k=0}^{\infty} F_k z^{-k}, \quad F_k \text{ an } m \times n \text{ matrix } (k = 0, 1, \dots) \quad *$$

is called an asymptotic expansion of the $m \times n$ matrix function $F(z)$ valid in a given range of values of $\arg z$, if for every fixed non-negative integer s , the expression

$$(1.4.2) \quad z^s \left\{ F(z) - \sum_{k=0}^s F_k z^{-k} \right\}$$

* The symbol " $\hat{=}$ " will here and henceforth mean a formal equality.

tends to the $m \times n$ zero matrix as $|z| \rightarrow \infty$ while $\arg z$ remains in the given range. When this is the case we shall write

$$(1.4.3) \quad F(z) \sim \sum_{k=0}^{\infty} F_k z^{-k}, \quad \text{or} \quad F(z) \sim \tilde{F}(z)$$

More generally, if $H(z)$ is a non-singular $m \times m$ matrix and

$$(1.4.4) \quad H(z)^{-1} F(z) \sim F_0 + F_1 z^{-1} + F_2 z^{-2} + \dots$$

we shall write

$$(1.4.5) \quad F(z) \sim H(z) \{F_0 + F_1 z^{-1} + F_2 z^{-2} + \dots\}.$$

The term $H(z) F_0$ is called the dominant term, or dominant matrix of the asymptotic representation (1.4.5) of $F(z)$.

The Borel-Ritt theorem (see e.g. [13] page B107) states that to any given formal series and any sector in the z -plane there exists a function $f(z)$ holomorphic in the given sector such that the given formal series is the asymptotic expansion of $f(z)$ as $|z| \rightarrow \infty$ in the sector.

1.5 The System of Differential Equations

In this thesis we are concerned with the system of differential equations

$$(1.5.1) \quad \frac{dW}{dz} = z^r A(z)W, \quad r \text{ integer}$$

from the point of view of asymptotic solutions. The matrix $A(z)$ is

$n \times n$ and holomorphic in a domain $\mathcal{D}$ extending to infinity in the z -plane. Moreover, for any positive integer s we have

$$(1.5.2) \quad A(z) = \sum_{k=0}^{s-1} A_k z^{-k} + A_s(z) z^{-s} \sim \sum_{k=0}^{\infty} A_k z^{-k}$$

where the elements of $A_s(z)$ are bounded for all sufficiently large $|z|$, z in $\mathcal{D}$. That is, the matrix $A(z)$ need merely have an asymptotic expansion for z in $\mathcal{D}$; we do not require it to have a convergent Taylor series expansion about $z = \infty$.

If $r \geq 0$ equation (1.5.1) is said to have an irregular singularity of rank $r + 1$ at infinity; if $r \geq -1$ equation (1.5.1) will be referred to as the singular equation.

An actual solution $W(z)$ of (1.5.1) is any non-singular $n \times n$ matrix of holomorphic functions which satisfies this equation for all z in $\mathcal{D}$. If $W(z)$ is an actual solution of (1.5.1) we shall also say that $W(z)$ constitutes a full set, that is, a linearly independent set, of (vector) solutions.

A formal solution $\tilde{W}(z)$ of (1.5.1) is a formal series of the type (1.4.3) or (1.4.5) with $m = n$ which, when substituted into (1.5.1) as if it were a convergent series, enables us to uniquely determine all the coefficients of the formal series when we equate equal powers of z^{-1} . If the dominant matrix of the formal series obtained in this manner is non-singular, we shall say that $\tilde{W}(z)$ constitutes a linearly independent set or a full set of n formal (vector) solutions.

1.6 A Review of the Literature Concerning Formal and Actual Solutions

Only a brief review of the literature relevant to the present work will be given.

In 1877 Thomé showed that the equation

$$(1.6.1) \quad y'' + p(z)y' + q(z)y = 0$$

$$p(z) = \sum_{k=0}^{\infty} p_k z^{-k}, \quad q(z) = \sum_{k=0}^{\infty} q_k z^{-k}, \quad |z| > \rho > 0$$

has formal solutions

$$(1.6.2) \quad \tilde{y}_j \sim e^{\omega_j z} z^{\rho_j} \sum_{k=0}^{\infty} a_{jk} z^{-k}; \quad a_{j0} \neq 0; \quad j = 1, 2.$$

The formal series on the right of (1.6.2) do not generally converge.

In 1885 Fabry [48] showed that there exists a linearly independent set of n formal solutions for the n^{th} order equation

$$(1.6.3) \quad P_n \frac{d^n y}{dz^n} + P_{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \dots + P_1 \frac{dy}{dz} + P_0 y = 0$$

where P_k ($k = 0, 1, \dots, n$) is an arbitrary polynomial in z .

In 1886 Poincaré [38], using Laplace transformations, showed that, if the degree of none of the polynomials P_k ($k = 0, 1, \dots, n$) exceeds that of P_n in (1.6.3), (i.e. the rank 1 case; for proof see e.g. [4] page 169), then the n formal solutions of (1.6.3) obtained by Fabry are asymptotic expansions of actual solutions of this equation.

Horn [39]^{*} extended the method of Poincaré to the case where the coefficients P_k ($k = 0, 1, \dots, n$) in (1.6.3) have a convergent power series expansion of the form

$$(1.6.4) \quad P_{n-k} = z^{kr} \sum_{\ell=0}^{\infty} P_{n-k,\ell} z^{-\ell}; \quad k = 0, 1, \dots, n$$

about $z = \infty$. He showed moreover that in the case when the roots of the characteristic equation

$$(1.6.5) \quad a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0 \quad (a_k = P_{k,0}; \quad k = 0, 1, \dots, n)$$

are distinct the system (1.6.3) with P_k ($k = 0, 1, \dots, n$) given by (1.6.4) has a linearly independent set of factorial series solutions of the form

$$(1.6.6) \quad y = \sum_{k=0}^{\infty} C_k \binom{1 - k + z/\gamma}{k}$$

where γ and C_k ($k = 0, 1, 2, \dots$) are constants. In his treatment of the equation (1.6.1) in [40] Horn uses the leading terms or partial sums of formal solutions to construct a differential equation which is in a certain sense close to the given equation when $|z|$ is large, and to prove, with the aid of Volterra integral equations, the existence of actual solutions of (1.6.1) for which the formal solutions (1.6.2) are

* This paper contains other references of the works of Horn.

asymptotic expansions.*

The existence of n linearly independent actual solutions of (1.6.3) for which the formal solutions of this equation are asymptotic expansions, in the case when z is real and the coefficients P_k ($k = 0, 1, \dots, n$) merely have asymptotic expansions, was proved by Sternberg [44] in 1920.

In 1909 [45] and again in 1913 [50] G.D. Birkhoff attempted to reduce the equation (1.5.1) in the case when $A(z)$ has a convergent power series expansion and $|r| \geq 0$ to the form

$$(1.6.7) \quad Y' = \left(z^r \sum_{j=0}^s c_j z^{-j} \right) Y$$

with $s = r + 1$, by making the transformation $X = B(z)Y$, where the n^2 elements of $B(z)$ are all regular for $|z|$ sufficiently large with the determinant $B(\infty)$ not zero. His proof is correct for the case when A_0 in (1.5.1) has distinct eigenvalues but it was pointed out by F.R. Gantmacher [49] that Birkhoff's Theorem (reproduced with proof in Ince [23] page 470) is false and that it will in general be necessary to take the integer s ((1.6.7) above) larger than $r + 1$.

In 1933 W.J. Trjitzinsky treated the equation (1.6.3) for the case when the coefficients P_k are holomorphic and need merely have an

*

For an earlier result along this line see Liouville, J. de Math (1) 2 (1838) p. 19 and (1) 3 (1838), p. 565.

asymptotic expansion (as $|z| \rightarrow \infty$) in some domain extending to infinity. He established the existence of a full set of actual solutions for which the formal solutions of this equation are asymptotic expansions. His theorems of a similar nature concerning the more general system (1.5.1) are also correct, although the existence of a linearly independent formal series solution of (1.5.1) had at this time not yet been fully established. Trjitzinsky [31] (1937) also extended the work of Horn [39] in obtaining convergent factorial series solutions in certain cases when the roots of the characteristic equation (1.6.5) are not necessarily distinct.

In 1935 Hokuhara proved the existence of a full set of formal solutions of (1.5.1). (Ill. [22]).

The paper of H.L. Turrittin [32] (1955) contains the most complete treatment to date of the reduction of the general system (1.5.1) to canonical form. This result is explicitly stated in Chapter III of this thesis. In his paper of 1955, Turrittin assumed the coefficient matrix in (1.5.1) to have a convergent power series expansion about $z = \infty$; this enabled him to replace certain of the divergent asymptotic series which represent solutions of (1.5.1) by convergent factorial series. He thereby extended the work of Trjitzinsky [31]. Two other papers of Turrittin extend the work of Birkhoff [46,45]. These concern (a) [34] the reduction of the system (1.5.1) to the Birkhoff canonical form - in this paper he proved the existence of the integer s (possibly larger than $r + 1$) in (1.6.7); and (b) [33] he replaced the $n \times n$ system (1.5.1) of rank $(r + 1)$ by an $n(r + 1) \times n(r + 1)$ system of rank 1.

The book [47] edited by C.H. Wilcox contains a more complete list of references than those mentioned above.

1.7 A Brief Review of the Literature Concerning Error Bounds

In 1890 H. Weber obtained error bounds for asymptotic approximations to the Hankel functions. By his method (summarized in [30] page 194) he used an integral representation for the Hankel function, together with solving the differential equation for the partial sum (which is used to approximate the Hankel function) of an asymptotic expansion.

In 1912 O. Blumenthal [21] obtained error bounds for the Liouville-Green (or WKB) approximation

$$(1.7.1) \quad W \doteq A q^{-\frac{1}{4}} \exp \left(\int u q^{\frac{1}{2}} dx \right) + B q^{-\frac{1}{4}} \exp \left(-u \int q^{\frac{1}{2}} dx \right)$$

for the solution to the equation

$$(1.7.2) \quad w'' = u^2 q(x) w$$

where u is a large parameter. His results are valid for x in an arbitrary finite interval.

In 1961 F.W.J. Olver [14] sharpened the results of Blumenthal, and generalized them for x a complex variable in some (possibly infinite) complex domain. In 1963-64 Olver [26,27] found error bounds for first approximations and asymptotic expansions in turning point problems. In 1964 he [35] wrote a general paper on error bounds for asymptotic

expansions, with application to Bessel functions of large argument. In 1964 he [6] wrote a paper on the asymptotic solution of second order differential equations having an irregular singularity of rank 1 with an application to Whittaker functions of large argument. In 1964 F.W.J. Olver and F. Stenger [7] found error bounds for asymptotic solutions of second order differential equations having an irregular singularity of arbitrary rank.

In all the above cases, Olver's procedure is to express the error term (i.e. the difference between an actual solution and the partial sum of a formal solution) in the form of a non-homogeneous second order differential equation and then to obtain accurate bounds for the solutions of this equation.

In his book of 1962 [29] H. Jeffreys gave a method of obtaining error terms for asymptotic expansions obtainable by use of Watson's Lemma ([29] page 14).

Richard [42] in 1963 found error bounds for the approximations $\sin \beta(x)$, $\cos \beta(x)$ $\left(\beta(x) = \int_0^x \sqrt{1 + \psi(\xi)} \, d\xi \right)$ to the solutions of the equation $y'' + \{1 + \psi(x) + r(x)\} y = 0$ where $\int_0^\infty |\psi'(x)| dx < \infty$, $\int_0^\infty |r(x)| dx < \infty$, and $\lim_{x \rightarrow \infty} [1 + \psi(x)] > 0$.

Useful results pertaining to error bounds for approximations to solutions of differential equations arise out of stability theory.

In particular we mention here Gronwall's [17] (or Bellman's [2]) inequality. The paper of Li Yue Shing contains (Bernoulli equation) norm bounds for the solution of $\frac{dX}{dz} = A(z)X + \phi(z, X)$ where $\|\phi(z, X)\| \leq P(z) \|X\| + Q(z) \|X\|^\alpha$, $\alpha \geq 0$. The paper of Willett and Wong [44] contains generalizations of Gronwall's inequality; this paper also contains a wealth of references on this inequality.

1.8 The Problem

An asymptotic expansion of a function $f(z)$ with respect to large values of the argument z can usually be shown to hold uniformly in some sector $\alpha + \delta \leq \arg z \leq \beta - \delta$, for some positive number δ in $0 < \delta < \frac{1}{2}(\beta - \alpha)$. This implies for example that for all z sufficiently large and in the sector $\alpha + \delta \leq \arg z \leq \beta - \delta$ we have

$$(1.8.1) \quad |f(z)/g(z) - a_0 - a_1 z^{-1} - \dots - a_k z^{-k}| < K |z|^{-k-1}$$

for some constant K depending only on δ . Such a bound gives one a very limited idea concerning the actual magnitude of error. It is known for example in the case of the Bessel functions that the expansion [35]

$$(1.8.2) \quad J_\nu(z) \sim \left(\frac{z}{\pi}\right)^{\frac{1}{2}} \left\{ \cos \varphi \left(1 - \frac{a_2}{z^2} + \frac{a_4}{z^4} + \dots\right) - \sin \varphi \left(\frac{a_1}{z} - \frac{a_3}{z^3} + \dots\right) \right\}$$

for fixed ν , where

$$(1.8.3) \quad \varphi = z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi, \quad a_k = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \dots (4\nu^2 - (2k-1)^2)}{8^k k!}$$

holds uniformly for large z in $|\arg z| \leq \pi - \delta$, $0 < \delta < \pi$.

The accuracy yielded by using a finite number of terms of each of the series on the right of (1.8.2) decreases steadily as $\arg z \rightarrow \pm \pi$.

That is, the constant K on the right of (1.8.1) becomes increasingly large as $\delta \rightarrow 0$.

Now in the case when a computer evaluates a function by its Taylor expansion he is warned of inaccuracies incurred near the boundary of the region of convergence since the magnitude of successive terms of the series decrease less slowly as the radius of convergence is approached. There is no such warning in the case of an asymptotic expansion because the rate of reduction of the magnitude of successive terms in the series is independent of the proximity of the boundary.

In deriving asymptotic expansions, "exponentially small" contributions to the final asymptotic series are discarded. For example [35] it is readily shown by repeated integration by parts that the integral

$$(1.8.4) \quad I(n) = \int_0^\pi \frac{\cos n t}{t^2 + 1} dt$$

has the Poincaré expansion

$$(1.8.5) \quad I(n) \sim (-1)^{n-1} \left(\frac{\lambda_1}{n^2} - \frac{\lambda_2}{n^4} + \frac{\lambda_3}{n^6} - \dots \right).$$

Evaluating (1.8.4) by direct numerical quadrature for $n = 10$ we obtain $I(10) = -0.0004558$ to seven decimals. On the other hand using (1.8.5)

for $n = 10$ we obtain $I(10) \sim (0.0005318 - 0.0000048 + 0.0000001 - \dots)$
 $= -0.0005271$, an incorrect result. If however we write (1.8.4) in the
 form

$$(1.8.6) \quad I(n) = \int_0^\infty \frac{\cos n t}{t^2 + 1} dt - \int_\pi^\infty \frac{\cos n t}{t^2 + 1} dt, \quad \text{we get}$$

$$\sim \frac{1}{2} \pi e^{-n} + (-1)^{n-1} \left(\frac{\lambda_1}{n^2} - \frac{\lambda_2}{n^4} + \frac{\lambda_3}{n^6} - \dots \right)$$

where the first integral on the right of (1.8.6) equals $\frac{1}{2} \pi e^{-n}$ and
 where the second has the same asymptotic expansion as (1.8.6). Using
 the "complete" expansion on the right of (1.8.6) to evaluate $I(10)$, we
 obtain the correct result - 0.0004558. The point $n = 10$ is not a
 special case; (1.8.6) gives a better result than (1.8.5) for all finite
 n . There is however no self-evident reason why this should be so.

Asymptotic analysts have long recognized that in general it
 is possible to achieve only limited (though often very high) accuracy
 when evaluating a function by its asymptotic expansion. In attempting
 to overcome this difficulty they have tried summing divergent series and
 obtaining "complete" asymptotic expansions.* The problem of finding
 convergent sums of divergent series solutions has not yet been completely
 solved even for the case where $A(z)$ in (1.5.1) has a convergent power
 series expansion about $z = \infty$. Finding convergent sums of divergent
 series solutions when $A(z)$ merely has an asymptotic expansion in certain
directions of infinity has not yet been solved.

* In the sense of Miller and Watson [24].

sectors of infinity has not at all been attempted.*

If however we demand somewhat less, namely, a bound on the difference between an actual solution and the partial sum of an asymptotic expansion for the actual solution, we obtain a problem which does not seem to be as difficult to solve. Moreover since in most cases it is much easier to evaluate a partial sum of an asymptotic expansion than the actual solution, or a convergent factorial series solution, such bounds are of considerable value.

1.9 Layout of the Thesis

In this thesis the method of Olver (see e.g. [6]) is extended to obtaining error bounds for a system of a first order equation of arbitrary rank. The method in [6] (or [7]) does not carry over to the arbitrary system directly, since a Liouville-Green (or WKB) transformation is not known for the general system of n first order equations. Other difficulties arise because of the appearance of more than two eigenvalues in the lead matrix of the general system, and because of two or more of these eigenvalues being the same.

As we have already mentioned, there exists a transformation which transforms the general system (1.5.1) to a canonical form (see Theorem 2.4.1). We shall find solutions and error bounds for solutions of the transformed system. It is possible that this transformed system has a regular point, a regular singular point or an irregular singular point at

*

See however, the paper of H.L. Turrittin in [47] for a different approach to this problem.

infinity. As an aid in overcoming difficulties we have started with the simplest case and progressively proceeded to the most difficult and complex case. Thus we considered first the case of a regular point.

The case of a regular singular point is more difficult in that we need to choose the canonical form (Section 2.2) so as to enable us to easily bound any particular j 'th error vector solution. Additional restrictions are required on the paths of integration in the Volterra integral equations defining the solutions.

When all eigenvalues of A_0 are distinct, it is possible, in the case of an irregular singular point to find a vector solution (both formal and actual) corresponding to each eigenvalue. The analysis for obtaining error bounds is considerably simpler in the case when the solution corresponds to an eigenvalue which is an extreme point^{*} of the smallest strictly convex closed polygon containing all the eigenvalues of A_0 , and is considered first. In this case the error can be expressed in the form of a single Volterra (vector) integral equation. In the case when the vector solution corresponds to an eigenvalue which is an interior point of the polygon, the error can be expressed in the form of a pair of simultaneous Volterra (vector) integral equations.

In all of the above cases we have assumed that end-points of integration can be chosen at infinity. This assumption is dropped when we consider the general case, which includes the above cases as well as all others.

* By extreme point we mean vertex.

In Chapter II of the thesis we transform the general system to canonical form, and explicitly construct formal solutions for the transformed system. We give an explicit construction of the transformation to canonical form for the case of a regular singular point and for the irregular singular case when all eigenvalues of the lead matrix are distinct. For Turrittin's theorem [32] concerning transformation of the general system to canonical form we state a slight modification which enables us to easily bound the solutions of integral equations.

In Chapter III of the thesis we obtain actual solutions and error bounds for formal partial sum approximations to actual solutions. The error bounds are given in two forms: as a norm bound and as a vector bound.

In Appendix A of the thesis we solve the general singular $n = 2$ case to illustrate the results of the thesis.

The proofs concerning existence and uniqueness of solutions of integral equations are contained in Appendix B.

Appendix C contains some extensions of Gronwall's [17] inequality to matrices, to Fredholm integral equations, and to simultaneous Volterra integral equations. These extensions are an aid in arriving at the results of theorems in Section 3.3.5 and 3.4.5 of the thesis.

Each chapter and each appendix begins with a short synopsis which summarizes its results.

CHAPTER II

TRANSFORMATIONS ON THE ORIGINAL SYSTEM AND FORMAL

SOLUTIONS FOR THE TRANSFORMED SYSTEM

The analysis is considerably simplified after transforming a given system of differential equations to canonical form. Besides transforming systems of differential equations to canonical form we also obtain formal solutions of the transformed system in this chapter.

Transformations which transform a given system to canonical form are explicitly described for the case of a regular singular point and for the case of an irregular singular point when the eigenvalues of the lead coefficient matrix are distinct. There is as yet no complete constructive proof of Turrittin's theorem concerning transformation of the very complicated general singular system to canonical form. We have however proved modifications of this theorem corresponding to our modifications of the usual canonical form for the case of a regular singular point. These modifications are an aid in bounding the solutions of integral equations in the next Chapter.

2.1 The Case of a Regular Point

In this case the given system of differential equations takes the special form

$$(2.1.1) \quad \frac{dW}{dz} = z^{-2} A(z)W$$

where for each arbitrary positive integer s

$$(2.1.2) \quad A(z) = \sum_{k=0}^{s-1} A_k z^{-k} + A_s(z) z^{-s} ; \quad A_k \text{ constant } (k = 0, 1, \dots)$$

and where the n^2 elements of $A_s(z)$ are bounded for z in some domain $\mathcal{D}$ extending to infinity. That is, we assume that the elements of $A_s(z)$ remain bounded as $|z| \rightarrow \infty$ in $\mathcal{D}$, and that $A(z)$ has the properties described in Section 1.5.

In the case of a regular point we need not make any preliminary transformations; the system (2.1.1)-(2.1.2) is ready for formal solution as it stands.

To obtain a formal solution we substitute

$$(2.1.3) \quad \tilde{W}(z) \hat{=} \sum_{k=0}^{\infty} U_k z^{-k}, \quad U_0 = I \quad *$$

into (2.1.1) with the coefficient matrix $A(z)$ formally expanded; i.e.

$$(2.1.4) \quad \sum_{k=0}^{\infty} (-)^k U_k z^{-k+1} \hat{=} \sum_{k=0}^{\infty} \sum_{\ell=0}^k A_{\ell} U_{k-\ell} z^{-k} \quad \text{or}$$

$$(2.1.5) \quad \sum_{k=0}^{\infty} \left[(k+1) U_{k+1} + \sum_{\ell=0}^k A_{\ell} U_{k-\ell} \right] z^{-k} \hat{=} 0$$

On equating equal powers of z to zero in (2.1.5) we obtain

Theorem 2.1.1 The system (2.1.1)-(2.1.2) has a formal independent series solution (2.1.3) with $U_0 = I$, and

$$(2.1.6) \quad U_{k+1} = - \sum_{\ell=0}^k A_{\ell} U_{k-\ell} / (k+1), \quad k = 0, 1, \dots$$

It is of course well known (see e.g. [4] Chapter III) that in the case when $A(z)$ has a convergent power series expansion in a neighborhood of $z = \infty$ then the formal series (2.1.3) converges in this same

* The sign " $\hat{=}$ " indicates a formal equality.

neighborhood of infinity and is an actual solution of this equation.

2.2 The Case of a Regular Singular Point

The given system now takes the form

$$(2.2.1) \quad \frac{dX}{dz} = z^{-1} A(z) X$$

where $A(z)$ is given by (2.1.2) and has the properties described in the previous section.

We shall reduce the system (2.2.1) to canonical form by a method which closely resembles that of H.L. Turritin [32] (pp. 31, 32 and 37) but with a slight modification allowing for actually carrying out the computations.

The normalizing transformation

$$(2.2.2) \quad X = T W$$

where T is a constant non-singular matrix is made in (2.2.1), changing (2.2.1) to the form

$$(2.2.3) \quad \frac{dW}{dz} = z^{-1} B(z) W \quad \text{where} \quad B(z) = T^{-1} A(z) T,$$

so that for each arbitrary positive integer s

$$(2.2.4) \quad B(z) = \sum_{k=0}^{s-1} B_k z^{-k} + B_s(z) z^{-s}$$

where $B_k = T^{-1} A_k T$ ($k = 0, 1, \dots$); $B_s(z) = T^{-1} A_s(z) T$. It follows that since the elements of $A_s(z)$ are bounded for all z in a particular domain extending to infinity, the elements of $B_s(z)$ are similarly bounded for z in this domain. It is further presumed that T has been chosen so that B_0 takes on the classical Jordan canonical form

$$(2.2.5) \quad B_0 = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & \cdots & 0 & M_\ell \end{bmatrix}$$

with M_i ($i = 1, 2, \dots, \ell$) a $\mu_i \times \mu_i$ $\left(\sum_{i=1}^{\ell} \mu_i = n\right)$ matrix of the form

$$(2.2.6) \quad M_i = \lambda_i I_{\mu_i} + J_{\mu_i}.$$

In (2.2.6) λ_i is a complex number, I_{μ_i} is a $\mu_i \times \mu_i$ unit matrix and J_{μ_i} is a $\mu_i \times \mu_i$ matrix with zeros everywhere except in its first subdiagonal where it may have ones or a mixture of zeros and ones.

If $0 \leq \operatorname{Re} \lambda_i < 1$, $-1 \leq \operatorname{Re} \lambda_j < 0$ and $\lambda_i - \lambda_j \neq 1$ ($i = 1, 2, \dots, \ell$, $i \neq j$) and if the transformation $W = T(z)X$ where $T(z)$ is a diagonal matrix with i 'th diagonal block I_{μ_i} ($i \neq j$) but with j 'th diagonal block $z^{-1} I_{\mu_j}$ completely fills B_1 with zeros we can then proceed directly to Theorem 2.2.1. If these conditions are not satisfied we may assume without loss of generality that

$$\operatorname{Re} \lambda_1 \leq \operatorname{Re} \lambda_2 \leq \cdots \leq \operatorname{Re} \lambda_\ell.$$

Draw vertical lines through the points $\lambda = k$ (k an integer) in the complex λ plane. If λ_i lies in the strip $k_i \leq \operatorname{Re} \lambda < k_i + 1$ we associate the integer k_i with λ_i ($i = 1, 2, \dots, \ell$). It is clear that $k_1 \leq k_2 \leq \dots \leq k_\ell$.

With $\kappa = k_\ell - k_1$, we next make the sequence of zero inducing transformations

$$(2.2.7) \quad Z_{k-1} = (I + z^{-k} T_k) Z_k \quad (k = 1, 2, \dots, \kappa); \quad Z_0 = W,$$

on (2.2.3). Thus for example, if

$$(2.2.8) \quad \frac{dZ_{k-1}}{dz} = z^{-1} \left[B_0 + B_1 z^{-1} + \dots + B_k z^{-k} + B_{k+1}(z) z^{-k-1} \right] Z_{k-1}$$

where, whenever possible all blocks $B_1, B_2, \dots, B_{k-1}$ have already been made to vanish, and $B_{k+1}(z)$ is defined in a domain $\mathcal{O}$ as on page 21 except that $|z|$ may need to be taken larger than before, to exclude the zeros of the determinant $\det(I + z^{-k} T_k)$, then

$$(2.2.9) \quad \frac{dZ_k}{dz} = z^{-1} \left(B_0 + B_1 z^{-1} + \dots + B_{k-1} z^{1-k} + C_k z^{-k} + B_{k+1}^*(z) z^{-k-1} \right) Z_k$$

where

$$(2.2.10) \quad C_k = B_k + B_0 T_k - T_k B_0 + k T_k$$

and where the elements of $B_{k+1}^*(z)$ are defined similarly as those of $B_{k+1}(z)$.

We divide the constant matrices C_k , B_k and T_k into smaller blocks in the same way that B_0 is subdivided in (2.2.5). After this subdivision is made we denote the i 'th row of blocks and the j 'th column of blocks respectively by C_{ijk} , B_{ijk} and T_{ijk} ($i, j = 1, 2, \dots, \ell$). Equation (2.2.10) for C_{ijk} then becomes

$$(2.2.11) \quad C_{ijk} = B_{ijk} + (\lambda_i I_{\mu_i} + J_{\mu_i}) T_{ijk} - T_{ijk} (\lambda_j I_{\mu_j} + J_{\mu_j}) + k T_{ijk};$$

if $\lambda_j - \lambda_i \neq k$, elements of T_{ijk} can be computed (for further details see [32] p. 33) starting with the right hand vector and calculating from top to bottom and in this manner successively computing the vectors from right to left so that $C_{ijk} = 0$. If $\lambda_j - \lambda_i = k$, C_{ijk} cannot in general be made to vanish, and we set $T_{ijk} = 0$.

Proceeding in this manner to $k = \kappa$, we fill all blocks of $B_1, B_2, \dots, B_{\kappa+1}$ with zeros, except those (i, j) 'th ($i < j$) for which $\lambda_j - \lambda_i = k = k_j - k_i$. We emphasize in particular that all blocks on and below the diagonal of $B_1, B_2, \dots, B_{\kappa}$ can be filled with zeros in this manner.

The root equalizing transformation

$$(2.2.12) \quad Z_{\kappa} = \begin{bmatrix} z^{k_1} I_{\mu_1} & & & & \\ & z^{k_2} I_{\mu_2} & & & \\ & & \ddots & & \\ & & & z^{k_{\ell}} I_{\mu_{\ell}} & \\ 0 & & & & \end{bmatrix} Y$$

applied to (2.2.9) with $k = \kappa$ changes this equation to

$$(2.2.13) \quad \frac{dY}{dz} = z^{-1} (B_0^* + B_1^{**}(z) z^{-1})Y$$

where B_0^* is zero below the diagonal. The matrices in the (i,i) 'th block of B_0^* (i.e. with respect to B_0 in (2.2.5)-(2.2.6)) are changed to $(\lambda_i - k_i)I_{\mu_i} + J_{\mu_i}$ ($i = 1, 2, \dots, \ell$) while a non-zero (i,j) 'th block ($i < j$) will be B_{ijk} (with respect to equation (2.2.9) with $k = \kappa$) if and only if $\lambda_j - \lambda_i = k$, a positive integer. Since the transformation (2.2.12) merely multiplies each block of $B(z)$ by some integral power of z , $B_1^{**}(z)$ is holomorphic in a domain $\mathcal{D}^{**}$ which coincides with the domain $\mathcal{D}$ in which $B_1(z)$ (page 21) is holomorphic for all $|z|$ larger than the largest zero of any of the determinants of the sequence of zero inducing transformations used. Moreover $B_1^{**}(z)$ has an expansion exactly similar to $B_1(z)$ defined in equation (2.2.4).

If some of the differences $\lambda_j - \lambda_i$, ($i, j = 1, 2, \dots, \ell$) were integers at the outset another normalizing transformation is applied to (2.2.13) to bring (2.2.13) to the form (2.2.3) where B_0 has the form (2.2.5) with the integer ℓ smaller than before. The new system, regardless of whether or not $\lambda_j - \lambda_i$ was an integer at the outset, will have all its eigenvalues in $0 \leq \operatorname{Re} \lambda_i < 1$.

For later convenience we apply another zero inducing transformation which makes B_1 zero, and lastly, we follow this up by a root equalizing transformation which puts the real part of a particular λ_j in $-1 \leq \operatorname{Re} \lambda_j < 0$ while for $i \neq j$, $0 \leq \operatorname{Re} \lambda_i < 1$. This last root

equalizing transformation takes the form $W = T(z)X$ where $T(z)$ has the form (2.2.12) with $k_i = 0$ ($i = 1, 2, \dots, \ell; i \neq j$) and $k_j = 1$.

Theorem 2.2.1 The system (2.2.1) can be transformed to the system (2.2.3) with $B(z)$ as in (2.2.4) and B_0 as in (2.2.5)-(2.2.6) by a finite sequence of normalizing, zero inducing and root equalizing transformations as described above. Moreover, corresponding to any fixed integer j in $[1, \ell]$ we can achieve $0 \leq \operatorname{Re} \lambda_i < 1$, $-1 \leq \operatorname{Re} \lambda_j < 0$ and $\lambda_i - \lambda_j \neq 1$ ($i = 1, 2, \dots, \ell; i \neq j$). The elements of $B(z)$ are holomorphic and bounded in a domain $\mathcal{D}$ which coincides with the domain in which $A(z)$ is holomorphic for all $|z|$ sufficiently large.

Theorem 2.2.2 The system (2.2.3) with $B(z)$ as described in the above theorem has a formal independent series solution

$$(2.2.14) \quad \tilde{W}(z) = \left(\sum_{k=0}^{\infty} U_k z^{-k} \right) \exp [B_0 \log z]$$

with $U_0 = I$ and U_k given by the unique solution of

$$(2.2.15) \quad k U_k + B_0 U_k - U_k B_0 = - \sum_{\ell=1}^k B_{\ell} U_{k-\ell}$$

Proof. Substituting (2.2.14) into (2.2.3), cancelling $\exp [B_0 \log z]$ on each side, using (2.2.4), expanding and equating equal powers of z we obtain (2.2.15). It is proved in detail by Turrittin in [32] p. 33 that with $U_0 = I$, (2.2.15) uniquely defines U_k , $k \geq 1$. Moreover, the j 'th column of blocks may be solved independently of the other blocks; the successive solution of columns in a particular column of blocks being carried out from right to left, and the elements of a particular column

being solved from top down.

It is well known (see e.g. [4] Chapter IV) that if $A(z)$ has a series expansion convergent in a neighborhood of $z = \infty$, then the series in round brackets on the right of (2.2.14) also converges in this same neighborhood of $z = \infty$, and hence in this case the formal solution (2.2.14) is an actual solution of (2.2.3).

We take this opportunity to make a remark concerning a possible economizing in computing successive coefficients. Suppose we are interested in knowing the coefficients of a particular vector solution - say the v 'th column $\tilde{W}_j^v(z)$ in the j 'th column of blocks $\tilde{W}_j(z)$ of $\tilde{W}(z)$.

Let J be a $\mu \times \mu$ matrix with zeros everywhere except in its first sub-diagonal where it has a full set of ones. Then by [5] page 32

$$(2.2.16) \quad \exp[J \log z] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \log z & 1 & 0 & \dots & 0 \\ \frac{(\log z)^2}{2!} & \log z & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{(\log z)^{\mu-1}}{(\mu-1)!} & \frac{(\log z)^{\mu-2}}{(\mu-2)!} & \dots & \dots & 1 \end{bmatrix}$$

Hence if J_{μ_j} has a one in its v 'th column as well as in its $v+1$ 'th, $\dots$, $(\mu-1)$ 'th but not in its μ 'th, where $\mu \leq \mu_j$ then

$$(2.2.17) \quad \tilde{W}_j^v(z) \hat{=} \sum_{k=0}^{\infty} \left\{ \sum_{s=v}^{\mu} U_{j,k}^s \frac{(\log z)^{s-v}}{(s-v)!} \right\} z^{-k}$$

where U_{jk}^s is the s 'th column vector in the j 'th column of blocks of U_k . Thus it is only necessary to compute the ν 'th, $\nu + 1$ 'th, $\dots$, μ 'th column at each stage, when computing the formal solution.

2.3 The Irregular Singular Case When the Eigenvalues of A_0 are Distinct

The given system now takes the form

$$(2.3.1) \quad \frac{dX}{dz} = z^r A(z) X$$

where r is a non-negative integer and where $A(z)$ has the properties described at the beginning of Section 2.1.

Theorem 2.3.1 If A_0 has n distinct eigenvalues then there
exist constant matrices $T_0, T_1, \dots, T_{r+1}$ such that when the trans-
formation

$$(2.3.2) \quad X = T(z) W = \left(\sum_{k=0}^{r+1} T_k z^{-k} \right) W$$

is applied to (2.3.1) we obtain

$$(2.3.3) \quad \frac{dW}{dz} = z^r C(z) W$$

where for each arbitrary positive integer s

$$(2.3.4) \quad C(z) = z^{r+1} \sum_{k=0}^{r+1} C_k z^{-k} + z^{-2} \sum_{k=0}^{s-1} B_k z^{-k} + B_s(z) z^{-s}.$$

and where the elements of $B_s(z)$ are holomorphic and bounded for all
 $|z|$ sufficiently large, $z \in \mathcal{O}$.

In (2.3.4) each matrix C_k ($k = 0, 1, \dots, r + 1$) is diagonal.

Proof. We shall prove this important theorem here (for the original proof see Birkhoff [46]) in order to furnish an explicit algorithm for computing the coefficients T_k ($k = 0, 1, \dots, r + 1$). By the analysis of the previous section we suppose that the transformation (2.2.2) has already been made and that the lead matrix $A_0 = C_0$ is in diagonal form with distinct diagonal elements. In this case we may take $T_0 = I$ in (2.3.2). On making the transformation (2.3.2) in (2.3.1) we obtain

$$(2.3.5) \quad \frac{dW}{dz} = z^r \left\{ T^{-1}(z) \left[A(z) T(z) - z^{-r} \frac{d T(z)}{dz} \right] \right\} W.$$

We next choose the $r + 1$ T_k 's ($k = 1, 2, \dots, r + 1$) so that the matrix

$$(2.3.6) \quad C(z) = T^{-1}(z) \left[A(z) T(z) - z^{-r} \frac{d T(z)}{dz} \right]$$

has the form (2.3.4). For this purpose we observe that in order to solve for the $(r + 1)$ T_k 's we need only consider the coefficients of $z^{-1}, z^{-2}, \dots, z^{-r-1}$ in the equation

$$(2.3.7) \quad T(z) C(z) = A(z) T(z) - z^{-r} \frac{d T(z)}{dz}.$$

Expanding (2.3.7) and equating equal powers of z^{-1} we obtain

$$(2.3.8) \quad \sum_{k=0}^{\ell} (T_{\ell-k} C_k - A_k T_{\ell-k}) = 0, \quad (\ell = 0, 1, \dots, r + 1)$$

With $\ell = 0$, (2.3.8) gives us $C_0 = A_0$. With $\ell = 1$ we have

$$(2.3.9) \quad T_1 C_0 - A_0 T_1 + C_1 - A_1 = 0.$$

Now for arbitrary T_1 , the diagonal part of $T_1 C_0 - A_0 T_1 = T_1 A_0 - A_0 T_1$ is zero. Thus $C_1 = \text{diag}(A_1)$. The off-diagonal elements of T_1 can be solved for from $\{T_1\}_{ij} (\{A_0\}_{jj} - \{A_0\}_{ii}) = \{A_1\}_{ij}$ under the assumption that the diagonal elements of A_0 are distinct. We set the undeterminable diagonal elements of T_1 to zero.

Suppose that we have solved for $C_0, C_1, \dots, C_\ell$ and the off-diagonal elements of $T_1, \dots, T_\ell$ and that we have set the diagonal elements of $T_1, \dots, T_\ell$ equal to zero. Then from (2.3.8) we have

$$(2.3.10) \quad T_{\ell+1} A_0 - A_0 T_{\ell+1} + \sum_{k=1}^{\ell} (T_{\ell+1-k} C_k - A_k T_{\ell+1-k}) + C_{\ell+1} - A_{\ell+1} = 0.$$

Again, the diagonal part of $T_{\ell+1} A_0 - A_0 T_{\ell+1}$ is zero. Thus the diagonal part of

$$(2.3.11) \quad \sum_{k=1}^{\ell} (T_{\ell+1-k} C_k - A_k T_{\ell+1-k}) + C_{\ell+1} - A_{\ell+1}$$

is also zero. This enables us to determine $C_{\ell+1}$. The off-diagonal elements of $T_{\ell+1}$ are determinable from

$$(2.3.12) \quad \{T_{\ell+1}\}_{ij} (\{A_0\}_{jj} - \{A_0\}_{ii}) + \left\{ \sum_{k=1}^{\ell} (T_{\ell+1-k} C_k - A_k T_{\ell+1-k}) \right\}_{ij} = 0,$$

($i \neq j$)

assuming the diagonal elements of A_0 to be distinct. The diagonal elements of $T_{\ell+1}$ are again not determinable and we set these equal to zero. This proves by induction that each T_k and C_k ($k = 0, 1, 2, \dots, r+1$) can be determined as described in the theorem.

On again expanding (2.3.7) and equating coefficients of z^{-r-1-k} , $k \geq 1$ we obtain

$$B_k = A_{k+r+2} - \sum_{\ell=1}^{\min(k, r+1)} (T_\ell B_{k-\ell} - A_{k+r+2-\ell} T_\ell) - \sum_{\ell=k+1}^{r+1} (T_\ell C_{k+\ell+2-\ell} - A_{k+r+2-\ell} T_\ell) + k T_k$$

for all $k \geq 0$ if we define $T_k = 0$ for $k > r+1$. Since the elements of $A(z)$, $T(z)$, $T^{-1}(z)$ are holomorphic and bounded for all $|z|$ sufficiently large, $z \in \mathcal{O}$, the elements of $B_s(z)$ are similarly holomorphic and bounded. This completes the proof of Theorem 2.3.1.

A desirable consequence of the previous theorem is that it simplifies considerably the construction of a formal solution (e.g. the proof to be given below is much shorter than that in Coddington and Levinson, [4] Chapter V).

Theorem 2.3.2 The system (2.3.3) where $C(z)$ has the canonical form (2.3.4) possesses a formal independent series solution matrix of the form

$$(2.3.13) \quad \tilde{W}(z) = \tilde{U}(z) \exp Q(z)$$

where $Q(z)$ is diagonal and has the form

$$(2.3.14) \quad Q(z) = \sum_{k=0}^r \frac{Q_k}{r+1-k} z^{r+1-k} + Q_{r+1} \ln z = \sum_{k=0}^r \frac{C_k z^{r+1-k}}{r+1-k} + C_{r+1} \ln z$$

and where

$$(2.3.15) \quad \tilde{U}(z) \triangleq I + \sum_{k=1}^{\infty} U_k z^{-k}$$

Proof. Substituting (2.3.13) into (2.3.3), cancelling $\exp Q(z)$, and equating coefficients of equal powers of z^{-1} , we find, that if we define $Q_j = 0$ for $j > r+1$, $U_j = 0$ for $j < 0$, then

$$(2.3.16) \quad \sum_{j=0}^{\min(k, r+1)} (U_{k-j} Q_j - C_j U_{k-j}) - (k - r - 1) U_{k-r-1} - \sum_{j=0}^{k-r-2} B_j U_{k-r-j-2} = 0 \quad (k = 0, 1, 2, \dots)$$

We shall prove that on choosing $U_0 = I$ and assuming $Q(z)$ diagonal, equation (2.3.16) thereafter uniquely determines each U_j and Q_j .

For, with $k = 0$ we have

$$(2.3.17) \quad Q_0 - C_0 = 0 \text{ which implies } Q_0 = C_0.$$

With $k = 1$,

$$(2.3.18) \quad U_1 C_0 - C_0 U_1 + Q_1 - C_1 = 0.$$

Now, if U_1 is an arbitrary $n \times n$ matrix, the diagonal part of $U_1 C_0 - C_0 U_1$ is zero. By assumption Q_1 and C_1 are diagonal and hence $Q_1 = C_1$. Since the diagonal elements of C_0 are distinct the off-diagonal elements of U_1 given by the solution of $\{U_1\}_{ij} (\{C_0\}_{jj} - \{C_0\}_{ii}) = 0$ are zero. The diagonal elements of U_1

are at this point not determinable and all we can say here is that $U_1 = D_1$ where D_1 is an arbitrary diagonal matrix.

Now suppose that for $1 \leq k \leq r$ we have $Q_k = C_k$, $U_k = D_k$ where D_k is an arbitrary diagonal matrix. Then solving for U_{k+1} and Q_{k+1} we have from (2.3.16),

$$(2.3.19) \quad U_{k+1} C_0 - C_0 U_{k+1} + \sum_{j=1}^k (D_{k+1-j} C_j - C_j D_{k+1-j}) + Q_{k+1} - C_{k+1} = 0$$

or, since diagonal matrices commute,

$$(2.3.20) \quad U_{k+1} C_0 - C_0 U_{k+1} + Q_{k+1} - A_{k+1} = 0$$

and on comparing equation (2.3.20) with (2.3.18) it follows by exactly similar arguments as those used for (2.3.18) that $Q_{k+1} = C_{k+1}$,

$U_{k+1} = D_{k+1}$ where D_{k+1} is an arbitrary diagonal matrix. Thus we have proved by induction that for any integer k in $0 \leq k \leq r+1$, $Q_k = C_k$ $U_k = D_k$ where D_k are yet undetermined diagonal matrices.

With $k = r+2$ we have from (2.3.16) and the above results,

$$(2.3.21) \quad U_{r+2} C_0 - C_0 U_{r+2} + \sum_{j=1}^{r+1} (D_{r+2-j} C_j - C_j D_{r+2-j}) - B_0 - D_1 = 0$$

or, since the sum $\sum$ on the left is a sum of commuting diagonal matrices,

$$(2.3.22) \quad U_{r+2} C_o - C_o U_{r+2} - B_o - D_1 = 0.$$

The diagonal part of $U_{r+2} C_o - C_o U_{r+2}$ is zero; hence $D_1 = U_1 = -\text{diag } B_o$. The off-diagonal elements of U_{r+2} are given by the solution of $\left\{U_{r+2}\right\}_{ij} \left(\left\{C_o\right\}_{jj} - \left\{C_o\right\}_{ii} \right) = \left\{B_o\right\}_{ij}$. The diagonal part, D_{r+2} of U_{r+2} cannot be determined at this point.

Now suppose that the matrices $U_1, U_2, \dots, U_\ell$ as well as the off-diagonal elements of the matrices $U_{\ell+1}, U_{\ell+2}, \dots, U_{r+\ell+1}$ have been uniquely determined, while the diagonal matrices D_k ($\ell + 1 \leq k \leq r + \ell + 1$) consisting of the diagonals of U_k ($\ell + 1 \leq k \leq r + \ell + 1$) are still undetermined. Then, from (2.3.16) we have

$$(2.3.23) \quad U_{r+\ell+2} C_o - C_o U_{r+\ell+2} + \sum_{j=1}^{r+1} (U_{r+2+\ell-j} C_j - C_j U_{r+2+\ell-j}) - \sum_{j=1}^{\ell+1} B_{j-1} U_{\ell+1-j} - (\ell + 1) U_{\ell+1} = 0.$$

On writing $U_j = \bar{U}_j + D_j$, $j > \ell$ we find, using the fact that diagonal matrices commute,

$$(2.3.24) \quad \bar{U}_{r+l+2} C_0 - C_0 \bar{U}_{r+l+2} + \sum_{j=1}^{r+1} (\bar{U}_{r+l+2-j} C_j - C_j \bar{U}_{r+l+2-j}) \\ - \sum_{j=1}^{l+1} B_{j-1} U_{l+1-j} - (l+1)(\bar{P}_{l+1} + D_{l+1})$$

and by arguments similar to those above, $\bar{U}_{r+l+2}$ as well as D_{l+1} is uniquely determined by (2.3.24).

Thus each U_j , $j \geq 1$ is uniquely determined. This completes the proof of Theorem 2.3.2.

Clearly the above procedure can be easily modified to compute a particular formal series solution vector independently of the others; the j 'th vector series solution being

$$(2.3.25) \quad \tilde{W}_j(z) = \tilde{U}_j(z) e^{q_j(z)}$$

where $\tilde{U}_j(z)$ is the j 'th vector in $\tilde{U}(z)$ and $q_j(z) = \{Q(z)\}_{jj}$. For later convenience we set

$$(2.3.26) \quad q_j(z) = \sum_{k=0}^r \frac{q_{jk}}{r+1-k} z^{r+1-k} + q_{j,r+1} \log z.$$

The representation (2.3.25) illustrates that there is a one-to-one correspondence between the eigenvalues of A_0 and the formal vector solutions.

2.4 The General Case

The following theorem due to H.L. Turrittin [32] concerns transformation of the system (2.4.1) with eigenvalues of A_0 not necessarily distinct to canonical form. By an exponential transformation in the statement of this theorem we mean a transformation of the form

$$(2.4.1) \quad W = e^{\lambda z^k} IX$$

where W and X are matrices, λ is a constant and k is an integer. The shearing transformation is defined in [32] page 39. We shall give an example of a shearing transformation for the $n = 2$ case in Appendix A.

Theorem 2.4.1 A given system of differential equations of the form

$$(2.4.2) \quad \frac{dX}{dt} = t^h A(t) X$$

where h is a non-negative integer, where for each arbitrary positive integer s

$$(2.4.3) \quad A(t) = \sum_{k=0}^{s-1} A_k t^{-k} + A_s(t) t^{-s} \hat{=} \sum_{k=0}^{\infty} A_k t^{-k}$$

and where the A_k 's are constant $n \times n$ matrices and the elements of $A_s(t)$ ($s = 1, 2, \dots$) are holomorphic and bounded in some domain $\mathcal{D}_t$ extending to infinity, can be transformed to the canonical system

$$(2.4.4) \quad \frac{dW}{dz} = z^r C(z) W$$

where

$$z^r C(z) = z^r \begin{bmatrix} C_1(z) & & & \\ & C_2(z) & & \\ & & \ddots & \\ & & & C_\ell(z) \end{bmatrix} + z^{-2} \begin{bmatrix} B_{11}(z) & B_{12}(z) & \cdots & B_{1\ell}(z) \\ B_{21}(z) & B_{22}(z) & \cdots & B_{2\ell}(z) \\ \cdots & \cdots & \cdots & \cdots \\ B_{\ell 1}(z) & B_{\ell 2}(z) & \cdots & B_{\ell\ell}(z) \end{bmatrix}$$

(2.4.5)

$$z^r C_i(z) = \sum_{k=0}^{r+1} c_{ik} z^{r-k} I_{\mu_i} + z^{-1} J_{\mu_i}$$

$$i = 1, 2, \dots, \ell, \quad \mu_1 + \mu_2 + \dots + \mu_\ell = n$$

$$B_{ij}(z) = \sum_{k=0}^{s-1} B_{ijk} z^{-k} + B_{ijs}(z) z^{-s}$$

by a finite sequence of normalizing, exponential, zero inducing, root

equalizing, and shearing transformations. In (2.4.4) r is an integer,

and $z = t^p$ for some positive integer p. The elements of the matrices

$B_{ij}(z)$ ($i, j = 1, 2, \dots, \ell$) are holomorphic and bounded for z in the domain

$\mathcal{D}$ which coincides with the domain $\mathcal{D}_t$ under the transformation $t^p = z$,

for all $|z|$ sufficiently large. If $r = -2$ the matrices $C_i(z)$ ($i = 1, 2, \dots, \ell$)

are identically zero. If $r > -2$, I_{μ_i} are $\mu_i \times \mu_i$ unit matrices, J_{μ_i}

are $\mu_i \times \mu_i$ square matrices with zeros or ones or a mixture of zeros and
ones on the first subdiagonal while all other elements of J_{μ_i} are zero.

No two of the matrices $C_i(z)$ ($i = 1, 2, \dots, \ell$) in (2.4.5) are identical.

In particular if $i \neq j$ and $c_{ik} = c_{jk}$ for $k = 0, 1, \dots, r$ then corresponding

to any particular fixed j ($j = 1, 2, \dots, \ell$), we have $0 \leq \operatorname{Re} c_{i,r+1} < 1$,

$-1 \leq \operatorname{Re} c_{j,r+1} < 0$ and $c_{i,r+1} - c_{j,r+1} \neq 1$ ($i = 1, 2, \dots, \ell$; $i \neq j$).

The above statement of Turrittin's theorem is in some aspects different from that given by Turrittin in [32]. In the first place Turrittin proves this theorem for the case when $A(t)$ has a convergent power series expansion about $t = \infty$ (actually in Turrittin's case $A(t)$ converges about $t = 0$ but this difference is immaterial). Secondly, Turrittin's original version of the last statement above is: In particular if $i \neq j$ and $c_{ik} = c_{jk}$ for $k = 0, 1, \dots, r$ then $c_{i,r+1} - c_{j,r+1}$ is not only not zero but is also not an integer.

We have already seen that the normalizing, zero inducing and root equalizing transformations preserve the form of the expansion of $A(t)$ as well as holomorphy and boundedness of the elements of $A(t)$ for $|t|$ sufficiently large, $t \in \mathcal{O}_t$. In addition the shearing transformation shifts the coefficients of t^{-k} in a manner similar to the root equalizing transformation and hence it also preserves these properties.

In order to arrive at our version of the last statement of Turrittin's theorem we start by assuming Turrittin's version and proceed by a method similar to that given on pages 25 and 26 of this chapter. We emphasize that the procedure which we shall now describe is necessary only if the last condition in our statement of Turrittin's theorem is not satisfied.

Consider the sub-set of integers $\{i\}$ of the set $\{i = 1, 2, \dots, l\}$ such that with $i \in \{i\}$, $j \in \{i\}$ and $i \neq j$, $c_{ik} - c_{jk} = 0$, $k = 0, 1, \dots, r$. Draw vertical lines through the points $\lambda = k$ (k an integer) in the complex λ plane. If $c_{i,r+1}$ lies in the strip $k_i \leq \operatorname{Re} \lambda < k_i + 1$ we associate the integer k_i with $c_{i,r+1}$. Without loss of generality we may assume

that $\{i\} = \{1, 2, \dots, \omega\}$, and that $k_1 \leq k_2 \leq \dots \leq k_\omega$. Let $\kappa = k_\omega - k_1$.

We shall next need $\kappa + 1$ ($n \times n$) matrices T_k ($k = 1, 2, \dots, \kappa + 1$) with each T_k being zero everywhere except in its leading $\omega \times \omega$ block where we shall define it presently. The sequence of zero inducing transformations (2.2.7) with T_k to be defined here and with κ (in (2.2.7)) replaced by κ (defined here) + 1, is now applied to (2.4.4). Thus for example if

$$\frac{dZ_{k-1}}{dz} = z^r [C_0 + C_1 z^{-1} + \dots + C_{r+1} z^{-r-1} + B_0 z^{-r-2} + \dots + B_k z^{-k-r-1} + \dots] Z_{k-1}$$

where the leading $\omega \times \omega$ blocks of $B_0, B_1, \dots, B_{k-1}$ have already been filled with zeros then

$$(*) \frac{dZ_k}{dz} = z^r [C_0 + C_1 z^{-1} + \dots + C_{r+1} z^{-r-1} + B_0 z^{-r-2} + \dots + B_k^* z^{-k-r-1} + \dots] Z_k$$

where

$$B_k^* = B_k - T_k C_{r+1} + C_{r+1} T_k + k T_k.$$

This equation is similar to (2.2.11); hence we can clearly choose T_k so that the leading $\omega \times \omega$ block of B_k^* is zero. It is thus clear that each leading $\omega \times \omega$ block of B_k ($k = 0, 1, \dots, \kappa + 1$) can in this manner be filled with zeros.

The root equalizing transformation (2.2.12) with $\ell = \omega$, with k_i ($i = 1, 2, \dots, \omega$; $i \neq j$) as defined here (above), but with k_j (in 2.2.12) replaced by k_j (as defined here) + 1, is now applied to (*) (above)

changing $c_{i,r+1}$ ($i = 1, 2, \dots, \omega$) to the form as described in our version of the last statement of Turrittin's theorem.

For details of proof of Theorem 2.4.1 we refer the reader to the paper cited above. We emphasize however that a constructive ("transparent" [47] page 7) proof of this extremely complicated theorem would be very desirable.

The following theorem is also proved by Turrittin [32]. The main purpose of presenting a proof here is to obtain a recurrence relation for the coefficients of the solution matrix and to furnish an explicit algorithm for computing these coefficients.

Theorem 2.4.2 The system of differential equations in canonical form (2.4.4)-(2.4.5) possesses a formal independent series solution matrix of the form

$$(2.4.6) \quad \tilde{W}(z) = \tilde{U}(z) e^{\gamma(z)}$$

where

$$\tilde{U}(z) = [\tilde{U}_{ij}(z)] \hat{=} \sum_{k=0}^{\infty} U_k z^{-k} \hat{=} \sum_{k=0}^{\infty} [U_{ijk}] z^{-k}$$

$$\tilde{U}_{ij}(z) \text{ is } \mu_i \times \mu_j, \quad U_{ij0} = \delta_{ij} I_{\mu_i}$$

$$(2.4.7) \quad Q(z) = [(I_{\mu_i} q_i(z) + J_{\mu_i} \log z) \delta_{ij}] = \sum_{k=1}^r \frac{Q_k}{r+1-k} z^{r+1-k} + Q_{r+1} \log z$$

$$q_i(z) = \sum_{k=0}^r \frac{q_{ik} z^{r+1-k}}{r+1-k} + q_{i,r+1} \log z \quad (i, j = 1, 2, \dots, \ell).$$

Proof of Theorem 2.4.2 Let us put

$$(2.4.8) \quad \begin{aligned} z^r [C_i(z) \delta_{ij}] &= \sum_{k=0}^{r+1} C_k z^{r+1-k} = z^r C(z) \\ [B_{ij}(z)] &= B(z) = \sum_{k=0}^{s-1} B_k z^{-k} + B_s(z) z^{-s} \end{aligned}$$

Hence if we substitute (2.4.6) into (2.4.4), use the fact that $Q(z)$ and $Q'(z)$ commute we obtain

$$(2.4.9) \quad \tilde{U}'(z) + \tilde{U}(z) Q'(z) \hat{=} \left(z^r C(z) + z^{-2} B(z) \right) \tilde{U}(z)$$

Expanding (2.4.9), using (2.4.7) and (2.4.8) and equating equal powers of z we obtain

$$(2.4.10) \quad \sum_{k=0}^{\min(\mu, r+1)} (C_k U_{\mu-k} - U_{\mu-k} Q_k) + (\mu-r-1) U_{\mu-r-1} + \sum_{k=0}^{\mu-r-2} B_k U_{\mu-r-2-k} = 0$$

($\mu = 0, 1, 2, \dots$) where, in order for this equation to be defined for each non-negative integer μ we put $U_k = 0$ if $k < 0$, $C_k = Q_k = 0$ if $k > r + 1$.

The proof of Theorem 2.4.2 will be complete if we prove the following lemma.

Lemma 2.4.1 Under the assumption that $Q(z)$ has the form given in (2.4.7) and $U_0 = I$, equation (2.4.10) thereafter uniquely determines Q_μ ($\mu = 0, 1, \dots, r+1$) and U_μ ($\mu = 1, 2, \dots$).

Proof of Lemma 2.4.1 The proof of the lemma is by induction.

With $\mu = 0$, (2.4.10) gives

$$(2.4.11) \quad C_0 - Q_0 = 0$$

which has the unique solution $Q_0 = C_0$. With $\mu = 1$, (2.4.10) becomes

$$(2.4.12) \quad C_1 - Q_1 + C_0 U_1 - U_1 C_0 = 0.$$

Now if U_1 is an arbitrary $n \times n$ matrix we know from the definition of C_0 that the (i,i) 'th ($i = 1, 2, \dots, \ell$) blocks of $C_0 U_1 - U_1 C_0$ consisting of $\mu_i \times \mu_i$ matrices are identically zero. Hence these blocks of $C_1 - Q_1$ must also be zero, and therefore $Q_1 = C_1$. Furthermore it follows that all (i,j) 'th blocks of U_1 for which $c_{i0} - c_{j0} \neq 0$ are uniquely determined and are zero. All (i,j) 'th blocks of U_1 for which $c_{i0} = c_{j0}$ cannot be solved for at this point. In particular it is not possible to solve for U_{ii1} ($i = 1, 2, \dots, \ell$).

Proceeding in this manner, let us suppose that we have just completed solving (2.4.10) for $\mu = k$ ($0 \leq k \leq r$); that we have found $Q_\mu = C_\mu$ ($\mu = 0, 1, \dots, k$) and that we have determined all blocks of the matrices U_μ ($\mu = 0, 1, \dots, k$) as far as possible. More precisely, we make the following assumptions for each (i,j) 'th block of U_μ ($i, j = 1, 2, \dots, \ell$; $\mu = 0, 1, \dots, k$). If $c_{i\mu} - c_{j\mu} = 0$ ($\mu = 0, 1, \dots, k-1$) then assume $U_{ij1}, U_{ij2}, \dots, U_{ijk}$ are arbitrary so far undetermined $\mu_i \times \mu_j$ matrices. On the other hand, if p is the smallest non-negative integer in $0 \leq p \leq k$

for which $c_{ip} - c_{jp} \neq 0$ we assume $U_{ijo} = U_{ij1} = \dots = U_{ij,k-p} = 0$ ($\mu_i \times \mu_j$), while $U_{ij,k+1-p}, U_{ij,k+2-p}, \dots, U_{ijk}$ are arbitrary so far undetermined $\mu_i \times \mu_j$ matrices. Under these assumptions let us proceed to $\mu = k + 1$.

For $\mu = k + 1$ equation (2.4.10) becomes

$$(2.4.13) \quad C_0 U_{k+1} - U_{k+1} C_0 + \dots + C_k U_1 - U_1 C_k + C_{k+1} - Q_{k+1} = 0.$$

If $U_1, U_2, \dots, U_{k+1}$ are arbitrary $n \times n$ matrices the (i,i) 'th ($i = 1, 2, \dots, \ell$) blocks of $C_\mu U_{k+1-\mu} - U_{k+1-\mu} C_\mu$ are zero ($\mu = 0, 1, \dots, k$). Hence these blocks of $C_{k+1} - Q_{k+1}$ must also be zero, and therefore $Q_{k+1} = C_{k+1}$. Let us incorporate this result in (2.4.13) and let us consider the (i,j) 'th block of (2.4.13) ($i, j = 1, 2, \dots, \ell$). Here we have

$$(2.4.14) \quad (c_{io} - c_{jo})U_{ij,k+1} + (c_{i1} - c_{j1})U_{ij,k} + \dots + (c_{ik} - c_{jk})U_{ij1} = 0.$$

If $c_{i\mu} - c_{j\mu} = 0$, $\mu = 0, 1, \dots, k$, then all we can say is that

$U_{ij1}, U_{ij2}, \dots, U_{ij,k+1}$ are so far undetermined arbitrary matrices. On the other hand, if p is the smallest non-negative integer in $0 \leq p \leq k$ for which $c_{ip} - c_{jp} \neq 0$, then by the above induction hypothesis (2.4.14) becomes

$$(2.4.15) \quad (c_{ip} - c_{jp}) U_{ij,k+1-p} = 0 \quad (\mu_i \times \mu_j)$$

and so $U_{ij,k+1-p} = 0$, under the assumption that $U_{ij,k+2-p}, \dots, U_{ij,k+1}$ are arbitrary $\mu_i \times \mu_j$ matrices.

It is thus clear at this point that Q_μ is uniquely determined for each $\mu = 0, 1, \dots, r+1$, and we may write the (i, j) 'th block of (2.4.10) in the form

$$(2.4.16) \quad \sum_{k=0}^{r+1} (q_{ij} - q_{jk}) U_{ij, \mu-k} + J_{\mu_i} U_{ij, \mu-r-1} - U_{ij, \mu-r-1} J_{\mu_j} + (\mu-r-1) U_{ij, \mu-r-1} \\ + \sum_{k=0}^{\mu-r-2} \sum_{s=1}^{\ell} B_{isk} U_{sj, \mu-r-2-k} = 0$$

For $\mu = r+2$, (2.4.16) becomes

$$(2.4.17) \quad \sum_{k=0}^{r+1} (q_{ik} - q_{jk}) U_{ij, r+2-k} + J_{\mu_i} U_{ij1} - U_{ij1} J_{\mu_j} + U_{ij1} + B_{ij0} = 0$$

If $q_{i\mu} - q_{j\mu} = 0$ ($\mu = 0, 1, \dots, r$) then for $U_{ij, r+2}$ an arbitrary $\mu_i \times \mu_j$ matrix

$$(2.4.18) \quad (1 + q_{i, r+1} - q_{j, r+1}) U_{ij1} + J_{\mu_i} U_{ij1} - U_{ij1} J_{\mu_j} + B_{ij0} = 0,$$

where under the assumption of Theorem 2.4.1 $q_{i, r+1} - q_{j, r+1} \neq -1$ if $i \neq j$.

In either case, whether $i = j$ or $i \neq j$ equation (2.4.18) is of the same form as equation (2.2.11) and hence it uniquely determines U_{ij1} .

On the other hand, suppose p is the smallest non-negative integer in

$0 \leq p \leq r$ for which $q_{ip} - q_{jp} \neq 0$. Then from the above analysis we know

that $U_{ij0} = U_{ij1} = \dots = U_{ij, r+1-p} = 0$ ($\mu_i \times \mu_j$). Thus if

$U_{ij, r+3-p}, \dots, U_{ij, r+2}$ are arbitrary $\mu_i \times \mu_j$ matrices

$$(2.4.19) \quad (q_{ip} - q_{jp}) U_{ij, r+2-p} + B_{ij0} = 0$$

and this equation uniquely determines $U_{ij,r+2-p}$ which has up to this point been defined only to be an arbitrary $\mu_i \times \mu_j$ matrix. Thus we see that the $\mu_i \times \mu_j$ matrix U_{ij1} is at this point uniquely determined for each pair of integers $(i,j : i,j = 1,2,\dots,\ell)$ (unless of course, only the j 'th column of blocks is desired, in which case it is unnecessary to compute any other column of blocks, since a particular column of blocks can be computed independently of any other), and hence U_1 has been completely and uniquely determined.

Proceeding in this manner, let us suppose that we have just completed solving (2.4.10) for $\mu = r + 2 + k$, $k \geq 0$; that we have uniquely determined $U_1, \dots, U_{1+k}$ and that we have determined all blocks of the other matrices U_{1+s} , $s > k$ as far as possible. More precisely, we make the following assumptions for each (i,j) 'th $U_{ij,1+\mu}$ of $U_{1+\mu}$, $0 \leq \mu \leq r + 2 + k$. If $q_{i\mu} - q_{j\mu} = 0$, ($\mu = 0,1,\dots,r$) then assume $U_{ij1}, \dots, U_{ij,r+2+k}$ are arbitrary so for undetermined $\mu_i \times \mu_j$ matrices. On the other hand, if p is the smallest non-negative integer in $0 \leq p \leq r$ for which $q_{ip} - q_{jp} \neq 0$ then assume $U_{ij1}, \dots, U_{ij,r+2+k-p}$ have been uniquely determined while $U_{ij,r+3+k-p}, \dots, U_{ij,r+2+k}$ are arbitrary so for undetermined $\mu_i \times \mu_j$ matrices. Under these assumptions we proceed to $\mu = r + 3 + k$.

With $\mu = r + 3 + k$ (2.4.16) is

$$\begin{aligned}
 (2.4.20) \quad & \sum_{s=0}^{r+1} (q_{is} - q_{js}) U_{ij,r+3+k-s} + J_{\mu_i} U_{ij,2+k} - U_{ij,2+k} J_{\mu_j} \\
 & + (k+2) U_{ij,2+k} + \sum_{s=0}^{k+1} \sum_{v=1}^{\ell} B_{ivs} U_{vj,k+1-s} = 0
 \end{aligned}$$

If $q_{is} - q_{js} = 0$ ($s = 0, 1, \dots, r$), then under the assumption that $U_{ij, r+3+k}$ is an arbitrary $\mu_i \times \mu_j$ matrix, by the above induction hypothesis, and the assumptions of Theorem 2.4.1, equation (2.4.20) is an equation in $U_{ij, k+2}$ of exactly the same type as (2.2.11) was in T_{ijk} , and thus (2.4.20) uniquely determines $U_{ij, k+2}$. On the other hand, suppose p is the smallest non-negative integer in $0 \leq p \leq r$ for which $q_{ip} - q_{jp} \neq 0$. Then assuming $U_{ij, r+4+k-p}, \dots, U_{ij, r+3+k}$ to be arbitrary $\mu_i \times \mu_j$ matrices, equation (2.4.20) may be written

$$(2.4.21) \quad (q_{ip} - q_{jp})U_{ij, r+3+k-p} = - \sum_{s=p+1}^{r+1} (q_{is} - q_{js})U_{ij, r+3+k-s} \\ - (k+2)U_{ij, k+2} - \sum_{s=0}^{k+1} \sum_{v=1}^{\ell} B_{ivs} U_{vj, k+1-s}$$

By the induction hypothesis, all matrices on the right of (2.4.21) are known, and hence $U_{ij, r+3+k-p}$ is uniquely determined.

Thus each U_{μ} ($\mu = 1, 2, \dots$) is uniquely determined by equation (2.4.10). This completes the proof of Lemma 2.4.1 and also of Theorem 2.4.2.

By a discussion analogous to that leading to equation (2.2.19) we obtain

$$(2.4.22) \quad \tilde{W}_j^v(z) \hat{=} \sum_{k=0}^{\infty} \left\{ \sum_{s=v}^{\mu} U_{jk}^s \frac{(\log z)^{s-v}}{(s-v)!} \right\} z^{-k} e^{q_j(z)}$$

for the v' 'th vector solution in the j' 'th column of blocks. In equation (2.4.22) we have assumed that J_{μ_j} has a one in its v' 'th, $v + 1'$ 'th, ..., $\mu - 1'$ 'th columns, but not in its μ' 'th, where $\mu \leq \mu_j$ and U_{jk}^s is the s' 'th ($s = 1, 2, \dots, \mu_j$) column vector in the j' 'th ($j = 1, 2, \dots, \ell$) column of blocks of the matrix U_k ($k = 1, 2, \dots$; Theorem 2.4.2). Thus to obtain a formal solution it is again only necessary to compute the μ' 'th, $\mu - 1'$ 'th, ..., v' 'th column of U_{jk} at each stage.

CHAPTER III

ERROR BOUNDS FOR FORMAL PARTIAL SUM APPROXIMATIONS TO ACTUAL SOLUTIONS

It is to be expected that the differential equations of the previous chapter possess actual solutions for which the formal solutions are asymptotic expansions (for proof of this statement see [8]).

In this chapter we generalize Olver's (see e.g. [35]) procedure for the second order equation in that we solve the differential equation for a partial sum of a formal series solution. Thus in setting up integral equations for the error vector we simultaneously prove the existence of actual solutions for which the formal series solutions obtained in the previous chapter are asymptotic expansions, as well as obtain error bounds for partial sum approximations to actual solutions.

The error bounds are given in two forms; as vector bounds and as norm bounds.

3.1 The Case of a Regular Point

3.1.1 The Differential Equation for an Approximation

We shall find an actual solution vector corresponding to the j 'th formal solution vector in (2.1.3). For this purpose we denote the j 'th column of U_k ($k = 0, 1, \dots$) as defined by Theorem 2.1.1 by U_{jk} .

The partial sum

$$(3.1.1) \quad \varphi_{jm}(z) = \sum_{k=0}^{m-1} U_{jk} z^{-k}$$

satisfies the differential equation

$$(3.1.2) \quad \frac{d \varphi_{jm}(z)}{dz} - z^{-2} A(z) \varphi_{jm}(z) = R_{jm}(z)$$

where on partially expanding $A(z)$ as in (2.1.2) and using (3.1.1) and (2.1.6) we find that

$$(3.1.3) \quad R_{jm}(z) = m U_{jm} z^{-m-1} - \sum_{k=0}^{m-1} A_{m-k}(z) U_{jk} z^{-m-2}.$$

Hence if a vector $W_j = W_j(z)$ satisfies (2.1.1) then the error vector

$$(3.1.4) \quad \epsilon_{jm}(z) = W_j(z) - \phi_{jm}(z)$$

satisfies the differential equation

$$(3.1.5) \quad \frac{d}{dz} \epsilon_{jm}(z) - z^{-2} A(z) \epsilon_{jm}(z) = -R_{jm}(z)$$

where $R_{jm}(z)$ is given in (3.1.3).

3.1.2 Matrix and Norm Error Bounds

Let $\mathcal{O}(z, \zeta)$ be the union of a set of points z which can be joined to ζ by a path $\mathcal{P}$ with the following properties:

1) $\mathcal{P}$ (with the exception of the point ζ if ζ is a boundary point of $\mathcal{O}$) lies wholly in $\mathcal{O}$;

2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $t'(s)$ is continuous and non-vanishing;

3)

$$(3.1.6) \quad \gamma_{\mathcal{P}}(t^{-k}) = \int |k t^{-k-1} dt| \quad (k = 1, 2, \dots)$$

is bounded.

Now if $\epsilon_{jm}(z)$ is a solution of

$$(3.1.7) \quad \epsilon_{jm}(z) = \int_{\xi}^z [t^{-2} A(t) \epsilon_{jm}(t) - R_{jm}(t)] dt$$

then $\epsilon_{jm}(z)$ also satisfies (3.1.5).

By the analysis of Appendix B $\mathcal{D}(z, \xi)$ is a domain, and the solution of (3.1.7) exists and is a unique vector of functions holomorphic in $\mathcal{D}(z, \xi)$.

By (3.1.3) we have, with $A_{ijs}(z) = \{A_s(z)\}_{ij}$, $U_{ijm} = \{U_m\}_{ij}$ and $R_{ijm}(z)$ the i 'th element of $R_{jm}(z)$, ($i = 1, 2, \dots, n$), that

$$(3.1.8) \quad |R_{ijm}(t)| \leq |m t^{-m-1}| |U_{ijm}| + (m+1) |t^{-m-2}| \gamma_{ijm}$$

where

$$(3.1.9) \quad (m+1) \gamma_{ijm} = \sup_{t \in \mathcal{D}} \left| \sum_{k=0}^{m-1} \sum_{s=1}^{\ell} A_{is, m-k}(t) U_{sjk} \right|.$$

We also define a matrix of non-negative elements B by

$$(3.1.10) \quad B = \sup_{t \in \mathcal{D}} [|\{A(t)\}_{ij}|]$$

to obtain, on denoting $|V_j|$ to be the vector of absolute values of the elements V_{ij} of the vector V_j , that

$$(3.1.11) \quad |\epsilon_{jm}(z)| \leq \int_{\xi}^z [|t^{-2}| B |\epsilon_{jm}(t)| + |m t^{-m-1}| U_{jm}| + (m+1) |t^{-m-2}| \gamma_{jm}] |dt|$$

Applying Lemmas C.1 and C.2 of Appendix C to (3.1.10) we obtain

Theorem 3.1.1 Let $\mathcal{D}(z, \zeta)$ be defined as above. Corresponding to
each formal solution vector $\tilde{W}_j(z)$ of (2.1.3) there exists an actual
solution vector $W_{jm}(z)$ of functions holomorphic in $\mathcal{D}(z, \zeta)$ and depending
on ζ and an arbitrary positive integer m such that

$$(3.1.12) \quad W_{jm}(z) = \sum_{k=0}^{m-1} U_{jk} z^{-k} + \epsilon_{jm}(z)$$

where

$$(3.1.13) \quad |\epsilon_{jm}(z)| \leq \exp \{B \gamma_{\mathcal{D}}(t^{-1})\} \{ \gamma_{\mathcal{D}}(U_{jm} t^{-m}) + \gamma_{jm} \gamma_{\mathcal{D}}(t^{-m-1}) \}.$$

In equation (3.1.12) U_{jk} is the j^{th} column vector of the matrix
 U_k ($k = 0, 1, \dots$) defined in Theorem 2.1.1, while in (3.1.13) the matrix
 B and the vector γ_{jm} are defined by equations (3.1.10) and (3.1.9)
respectively.

Similarly, to obtain a norm bound for $\epsilon_{jm}(z)$ above, let

$$(3.1.14) \quad B = \sup_{t \in \mathcal{D}} \|A(t)\|, \quad (m+1) \gamma_{jm} = \sup_{t \in \mathcal{D}} \left\| \sum_{k=0}^{m-1} A_{m-k}(t) U_{jk} \right\|$$

where any one of the definitions of norms in Chapter I may be used.

Theorem 3.1.2 A corresponding norm bound $\epsilon_{jm}(z)$ in Theorem 3.1.1
is

$$(3.1.15) \quad \|\epsilon_{jm}(z)\| \leq \exp \{B \gamma_{\mathcal{D}}(t^{-1})\} \{ \|U_{jm}\| \gamma_{\mathcal{D}}(t^{-m}) + \gamma_{jm} \gamma_{\mathcal{D}}(t^{-m-1}) \}$$

where B and γ_{jm} are defined by equation (3.1.14).

3.2 The Case of a Regular Singular Point

3.2.1 The Differential Equation for an Approximation

With reference to Theorems 2.2.1 and 2.2.2 we shall find error bounds for an actual solution vector of (2.2.3) corresponding to the v 'th formal solution vector (2.2.17) in the j 'th column of blocks of formal solutions.

The partial sum

$$(3.2.1) \quad \varphi_{jm}(z) = \left(\sum_{k=0}^{m-1} U_{jk} z^{-k} \right) \exp [(\lambda_j I_{\mu_j} + J_{\mu_j}) \log z]$$

satisfies the differential equation

$$(3.2.2) \quad \frac{d}{dz} \varphi_{jm}(z) - z^{-1} B(z) \varphi_{jm}(z) = R_{jm}(z) \exp [(\lambda_j I_{\mu_j} + J_{\mu_j}) \log z].$$

On expanding the left hand side of (3.2.2) we obtain

$$(3.2.3) \quad \begin{aligned} R_{jm}(z) = & [U_{j0} (\lambda_j I_{\mu_j} + J_{\mu_j}) - B_0 U_{j0}] z^{-1} \\ & + \sum_{k=1}^{m-1} \{ -k U_{jk} + U_{jk} (\lambda_j I_{\mu_j} + J_{\mu_j}) - B_0 U_{jk} - \sum_{\ell=1}^k B_{\ell} U_{k-\ell} \} z^{-k-1} \\ & - \left(\sum_{\ell=1}^m B_{\ell} U_{m-\ell} \right) z^{-m-1} - \left(\sum_{s=0}^{m-1} B_{m+1-s}(z) U_{js} \right) z^{-m-2} \end{aligned}$$

But by equation (2.2.15) we obtain

$$(3.2.4) \quad R_{jm}(z) = [m U_{jm} + B_o U_{jm} - U_{jm} (\lambda_j I_{\mu_j} + J_{\mu_j})] z^{-m-1} \\ - \left(\sum_{s=0}^{m-1} B_{m+1-s}(z) U_{js} \right) z^{-m-2}$$

Thus, if a block of μ_j vectors $W_j(z)$ satisfies (2.2.3) the error block of vectors

$$(3.2.5) \quad \eta_{jm}(z) = [W_j(z) - \varphi_{jm}(z)]$$

satisfies differential equation

$$(3.2.6) \quad \frac{d}{dz} \eta_{jm}(z) - z^{-1} B(z) \eta_{jm}(z) = - R_{jm}(z) \exp [(\lambda_j I_{\mu_j} + J_{\mu_j}) \log z]$$

where $R_{jm}(z)$ is given in (3.2.4) above.

3.2.2 Matrix and Norm Error Bounds

With $\mathcal{O}$ the domain in which $B(z)$ (Theorem 2.2.1) is holomorphic, let $\mathcal{O}^* = \mathcal{O} - \{0\} - \{\infty\}^\dagger$. We define a region $\mathcal{O}(z, \zeta)$ by the union of the set of points z ($\neq \zeta$) which can be joined to ζ by a path $\mathcal{P}$ with the following properties:

- 1) Except for ζ , $\mathcal{P}$ lies wholly in $\mathcal{O}^*$;
- 2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ with $\frac{dt(s)}{ds}$ continuous and non-vanishing for each;
- 3) τ/t remains bounded for every pair of points t, τ in the order ζ, t, τ, z on $\mathcal{P}$;

[†] By e.g. $\{\infty\}$ we mean the set $\infty e^{i\theta}$, θ real.

$$4) \quad \gamma_{\mathcal{P}}(t^{\eta-1}) = \int_{\mathcal{P}} |(\eta - 1) t^{\eta-2} dt|$$

is bounded, where η may be taken to be zero if $J_{\mu_j} = 0$, but otherwise η is an arbitrary positive number less than 1.

Set $B_{oj} = \lambda_j I_{\mu_j} + J_{\mu_j}$. Equation (3.2.6) may be written in the form

$$(3.2.7) \quad \frac{d}{dz} \eta_{jm}(z) - z^{-1} B_o \eta_{jm}(z) = z^{-2} B_1(z) \eta_{jm}(z) - R_{jm}(z) e^{B_{oj} \log z}.$$

By our choice of a canonical form in Section 2.2. and our assumptions on the path above $\int_{\mathcal{P}} |\rho dt|$ exists, where ρ is any element of the matrix $R_{jm}(t) \exp \{B_{oj} \log t\}$. Hence if $\eta_{jm}(z)$ is a solution of

$$(3.2.8) \quad \eta_{jm}(z) = \int_{\zeta}^z \exp \{B_o \log \frac{z}{t}\} \{t^{-2} B_1(t) \eta_{jm}(t) - R_{jm}(t) e^{B_{oj} \log t}\} dt$$

where the integral is taken along one of the paths $\mathcal{P}$ of the family described above, $\eta_{jm}(z)$ also satisfies (3.2.7).

Let us put $\eta_{jm}(z) = \epsilon_{jm}^{\lambda_j}(z) z^{\lambda_j}$, and let us consider the v 'th column vector $\epsilon_{jm}^v(z)$ of $\epsilon_{jm}(z)$. With reference to the discussion surrounding equation (2.2.17) the integral equation for $\epsilon_{jm}^v(z)$ takes the form

$$(3.2.9) \quad \epsilon_{jm}^v(z) = \int_{\zeta}^z \exp \left\{ (B_o - \lambda_j I) \log \left(\frac{z}{t} \right) \right\} \left\{ t^{-2} B_1(t) \epsilon_{jm}^v(t) \right. \\ \left. - \sum_{s=v}^{\mu} R_{jm}^s(t) \frac{(\log t)^{s-v}}{(s-v)!} \right\} dt$$

where $R_{jm}^s(t)$ is the s 'th column vector ($s=v, v+1, \dots, \mu$) in $R_{jm}(t)$.

Now

$$(3.2.10) \quad \frac{d}{dt} \left(e^{-B_o \log t} U_{jm} t^{-m} e^{B_{oj} \log t} \right) \\ = - e^{-B_o \log t} [m U_{jm} + B_o U_{jm} - U_{jm} B_{oj}] t^{-m-1} e^{B_{oj} \log t},$$

thus the first term of $R_{jm}(z)$ in (3.2.4) is in a form suitable for bounding. The contribution to the v 'th error vector of the second term in (3.2.4) is

$$(3.2.11) \quad \int_{\zeta}^z \exp [(B_o - \lambda_j I) \log \left(\frac{z}{t} \right)] \sum_{p=0}^{m-1} \sum_{s=v}^{\mu} B_{m+1-p}(t) U_{jp}^s \frac{(\log t)^{s-v}}{(s-v)!} t^{-m-2} dt$$

To obtain a vector bound we choose a fixed number η_1 where

$\eta_1 = 0$ if $J_{\mu_i} = 0$ ($i = 1, 2, \dots, l$) but is otherwise an arbitrary positive number less than 1, and set

$$(3.2.12) \quad (m+1-\eta_1) \gamma_{ij}^{\omega v} \\ = \sup_{t, \tau \in \mathcal{P}} \left| t^{-\eta_1} \left\{ \exp [(B_o - \lambda_j I) \log \left(\frac{\tau}{t} \right)] \sum_{p=0}^{m-1} \sum_{s=v}^{\mu} B_{m+1-p}(\tau) U_{jp}^s \frac{(\log \tau)^{s-v}}{(s-v)!} \right\}_i \right|$$

where t and τ are located on $\mathcal{P}$ in the order ζ, t, τ, z , and where

$\{V_j\}_i^{\omega}$ indicates the ω 'th element of the group of elements labelled i (corresponding to the i 'th row of blocks of $B(z)$) of the vector V_j . We

define γ_j^v such that $\{\gamma_j^v\}_i^{\omega} = \gamma_{ij}^{\omega v}$.

Similarly we fix a number η where $\eta = 0$ if $J_{\mu_j} = 0$ but otherwise η is any positive number less than 1, and for t and τ in the order ζ, t, τ, z on $\mathcal{P}$ we define

$$(3.2.13) \quad (1 - \eta) B = \left[\sup_{t, \tau \in \mathcal{P}} \left| \left\{ t^{-\eta} \exp [(B_0 - \lambda_j I) \log \left(\frac{\tau}{t} \right)] B_1(t) \right\}_{ij}^{ps} \right| \right]$$

where $\{A\}_{ij}^{ps}$ is the (p,s) 'th element of the (i,j) 'th block of A .

By Section B.1 of Appendix B $\mathcal{Q}(z, \zeta)$ is a domain (actually a Riemann surface) and the solution to (3.2.9) exists and is a unique vector of functions holomorphic in $\mathcal{Q}(z, \zeta)$. Applying the above inequalities to (3.2.9) we have

$$(3.2.14) \quad |\epsilon_{jm}^\nu(z)| \leq \int_{\zeta}^z [(1 - \eta) |t^{\eta-2}| |B| |\epsilon_{jm}^\nu(t)| + \left| \frac{d}{dt} \sum_{s=\nu}^{\mu} U_{jm}^s \frac{(\log t)^{s-\nu}}{(s - \nu)!} t^{-m} \right| + (m + 1 - \eta_1) \gamma_j^\nu |t^{-m-1+\eta_1}|] |dt| .$$

Applying Lemmas C.1 and C.2 of Appendix C to (3.1.13) we obtain

Theorem 3.2.1 Let $\mathcal{Q}(z, \zeta)$ be defined as above. Corresponding to the ν 'th formal solution vector in the j 'th column of blocks of formal solutions as described by Theorem 2.2.2. for the equation (2.2.3)-(2.2.4), there exists an actual solution vector $W_{jm}^\nu(z)$ of functions holomorphic in $\mathcal{Q}(z, \zeta)$ depending on ζ and an arbitrary positive integer m , such that

$$(3.2.15) \quad W_{jm}^\nu(z) = \left[\sum_{p=0}^{m-1} \sum_{s=\nu}^{\mu} U_{jp}^s \frac{(\log z)^{s-\nu}}{(s - \nu)!} z^{-p} + \epsilon_{jm}^\nu(z) \right] z^{\lambda_j}$$

where

$$(3.2.16) \quad |\epsilon_{jm}^v(z)| \leq \exp \{B \gamma_{\mathcal{P}}(t^{\eta-1})\} \left\{ \gamma_{\mathcal{P}} \left(\sum_{s=v}^{\mu} U_{jm}^s \frac{(\log t)^{s-v}}{(s-v)!} t^{-m} \right) + \gamma_j^v \gamma_{\mathcal{P}}(t^{-m-1+\eta_1}) \right\}$$

for each z in $\mathcal{Q}(z, \zeta)$. The integers μ and ν , and the vectors U_{jm}^s are defined as in equation (2.2.17). If each $J_{\mu_i} = 0$ ($i = 1, 2, \dots, \ell$) then the numbers η and η_1 may be taken to be zero; if $J_{\mu_i} \neq 0$ then η may be taken to be zero; otherwise η and η_1 are arbitrary positive numbers less than 1. The $n \times n$ matrix $B = B(\eta)$ and the $n \times 1$ vector $\gamma_j^v = \gamma_j^v(\eta_1)$ are defined through equations (3.2.13) and (3.2.12) respectively.

To obtain a norm error bound, we first choose any of the definitions of norms given in Chapter I. Next, for η and η_1 defined as for Theorem 3.2.1 and t and τ in the order ζ, t, τ, z on $\mathcal{P}$, we define

$$(3.2.17) \quad (m+1-\eta_1) \gamma_j^v = \sup_{t, \tau \in \mathcal{P}} \|t^{-\eta_1} \exp [(B_0 - \lambda_j I) \log \left(\frac{\tau}{t} \right)] \sum_{p=0}^{m-1} \sum_{s=v}^{\mu} B_{m+1-p}^{(t)} U_{jp}^s \frac{(\log t)^{s-v}}{(s-v)!}\|$$

and

$$(3.2.18) \quad B = \sup_{t, \tau \in \mathcal{P}} \|t^{-\eta} \exp [(B_0 - \lambda_j I) \log \left(\frac{\tau}{t} \right)] B_1(t)\|$$

Theorem 3.2.2 A norm bound for $\epsilon_{jm}^v(z)$ in Theorem 3.2.1 is given
by

$$(3.2.19) \quad \|\epsilon_{jm}^v(z)\| \leq \exp \left\{ B \gamma_{\varphi}(t^{\eta-1}) \right\} \left\{ \sum_{s=v}^{\mu} \|U_{jm}^s\| \gamma_{\varphi} \left(\frac{(\log t)^{s-v}}{(s-v)!} t^{-m} \right) + \gamma_j^v \gamma_{\varphi}(t^{-m-1+\eta_1}) \right\}$$

where γ_j^v and B are positive numbers defined by (3.2.17) and (3.2.18) respectively.

3.3 The Irregular Singular Case when the Eigenvalues of A_0 are Distinct

3.3.1 The Differential Equation for an Approximation

We shall obtain an actual solution vector $W_j(z)$ corresponding to the j 'th formal solution vector $\tilde{W}_j(z)$ as defined by (2.3.25). The procedure simultaneously gives us a bound on the difference between an actual solution and the sum of the first m terms of a formal solution.

We denote the j 'th column of U_k ($k = 0, 1, \dots$) as defined by Theorem 2.3.2 by U_{jk} . Let the partial sum

$$(3.3.1) \quad \varphi_{jm}(z) = \left(\sum_{k=0}^{m-1} U_{jk} z^{-k} \right) e^{q_j(z)}$$

satisfy the differential equation

$$(3.3.2) \quad \frac{d \varphi_{jm}(z)}{dz} - z^r C(z) \varphi_{jm}(z) = R_{jm}(z) e^{q_j(z)}$$

where $C(z)$ is defined by (2.3.4); that is, we use (3.3.2) to define the vector $R_{jm}(z)$.

Expanding (3.3.1), using equations (2.3.26) and (2.3.16) we obtain

$$(3.3.3) \quad R_{jm}(z) = - \sum_{\mu=0}^{r+m+1} \left\{ \sum_{k=0}^{\min(\mu, r+1)} (Q_k - Iq_{jk}) U_{j, \mu-k}^+ + (\mu-r-1) U_{j, \mu-r-1}^+ \right. \\ \left. + \sum_{k=0}^{\mu-r-2} B_k U_{j, \mu-r-2-k} \right\} z^{r-\mu} - \sum_{k=0}^{m-1} \left(B_{m-k}(z) U_{jk} \right) z^{-m-2}$$

where $U_{jk}^+ = \begin{cases} U_{jk} & \text{if } 0 \leq k \leq m-1; \\ 0 & \text{otherwise} \end{cases}$.

Consider the i 'th element $R_{ijm}(z)$ of $R_{jm}(z)$. If $i = j$ then, using (2.3.16) once more we obtain

$$(3.3.4) \quad R_{jjm}(z) = - \left\{ \sum_{k=0}^{m-1} (B_k U_{j, m-1-k}) z^{-m-1} + B_{m-k}(z) U_{jk} z^{-m-2} \right\}_j$$

where $\{V_j\}_i$ indicates the i 'th element of the vector V_j . But on again using (2.3.16) we have

$$(3.3.5) \quad R_{jjm}(z) = m U_{jjm} z^{-m-1} - \left\{ \sum_{k=0}^{m-1} (B_{m-k}(z) U_{jk}) z^{-m-2} \right\}_j$$

If $i \neq j$ we define

$$(3.3.6) \quad q_{ij}(z) = q_i(z) - q_j(z)$$

and again make use of (2.3.16) in (3.3.3) to obtain

$$\begin{aligned}
 (3.3.7) \quad R_{ijm}(z) &= q'_{ij}(z)U_{ijm} z^{-m} - z[q'_{ij}(z) - (q_{io} - q_{jo})z^r]U_{ijm} z^{-m-1} \\
 &- \sum_{\mu=m+1}^{m+r+1} \left[\sum_{k=\mu+1-m}^{\min(\mu, r+1)} (q_{ik} - q_{jk})U_{ij, \mu-k} + (\mu-r-1)U_{ij, \mu-r-1} \right. \\
 &\left. + \sum_{k=0}^{\mu-r-2} \left\{ B_k U_{j, \mu-r-2-k} \right\}_i \right] z^{r-\mu} - \sum_{k=0}^{m-1} \left\{ B_{m-k}(z)U_{jk} \right\}_i z^{-m-2}
 \end{aligned}$$

3.3.2 Two Possible Cases

In this section we shall illustrate that it will in all cases be possible to express the error vector by at most a simultaneous pair of Volterra vector integral equations^{*}. The error bounds we shall obtain are considerably sharper in the case when we can express the error by a single Volterra vector integral equation than in the case when we require a simultaneous pair of Volterra vector integral equations to express the error vector. Thus it is desirable whenever possible to use a single Volterra vector integral equation.

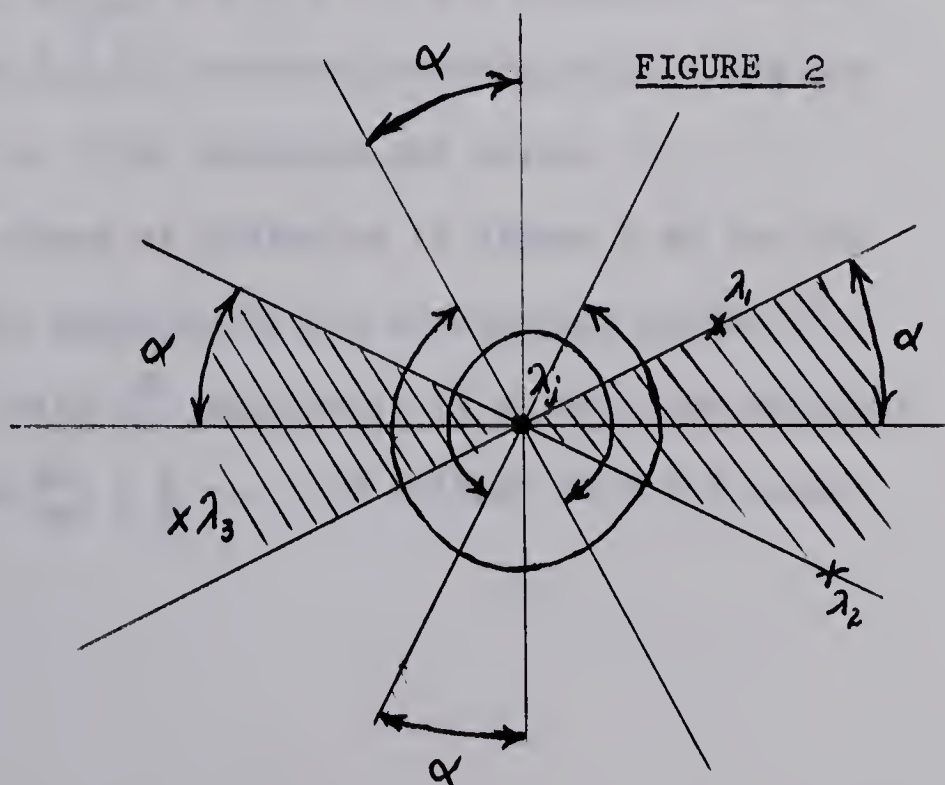
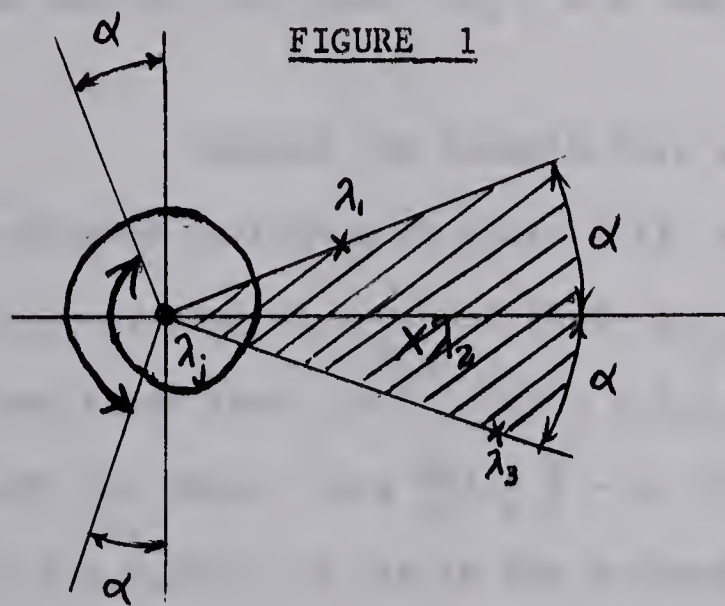
We have already noted at the end of Section 2.3 that by our construction of formal solutions there is a one-to-one correspondence between the eigenvalues of C_0 and the vector solutions. Thus whenever it is possible to express the error vector by a single Volterra vector

*

We assume throughout this section that end-points of integration be taken at infinity.

integral equation we shall refer to the corresponding eigenvalue as an extreme eigenvalue; otherwise we shall call it an interior eigenvalue.

Consider the eigenvalues of C_0 to be represented as points in the complex plane, and to be enclosed there by a smallest closed convex polygon Π . If an eigenvalue of C_0 is an extreme point (i.e. a vertex) of the polygon it will always be an extreme eigenvalue; if it is an interior point but not a boundary point it will always be an interior eigenvalue; if it is a boundary point but not an extreme point, then to determine whether it is an extreme eigenvalue or an interior eigenvalue it is necessary to also examine the eigenvalues of C_1 .



The polygon Π and the differences between the various eigenvalues of A_0 are left invariant under the transformations $W = X e^{\lambda z^{r+1}}$, $W = T(z)X$ ($T(z)$ as in Theorem 2.3.1), $z = \omega \xi$, ω a constant and $|\omega| = 1$. Thus we may assume without loss of generality that the vector solution of interest corresponds to the zero eigenvalue. Moreover, in the case of an extreme eigenvalue, e.g. when zero is an extreme point of the polygon Π , we may assume that all other eigenvalues of A_0 are within the closed sector $|\arg \lambda| \leq \alpha \leq \frac{1}{2} \pi$ and that there is at least one eigenvalue on each of the rays $\arg \lambda = \alpha$ and $\arg \lambda = -\alpha$. In the case of an interior eigenvalue we assume that no eigenvalue (other than the zero eigenvalue) is located on the imaginary axis; that there are some eigenvalues of A_0 in each of the sectors $|\arg \lambda| \leq \alpha$ and $|\pi - \arg \lambda| \leq \alpha$, $0 < \alpha < \frac{1}{2} \pi$ and that there is at least one eigenvalue on each of the lines $\arg \lambda = \alpha$ and $\arg \lambda = -\alpha$.

Suppose for example that we have the extreme eigenvalue case indicated in Figure 1; that $C(z)$ is regular (holomorphic) for all $|z| > \rho$; that $r = 0$ and that $q_{i,r+1} = q_{i1} = 0$ ($i = 1, 2, 3, 4$). It is then clear that $|e^{\lambda_i z}|$ ($i = 1, 2, 3, 4$) increases monotonically along any path for which $|\arg \frac{dz}{dx}| \leq \frac{\pi}{2} - \alpha$. Thus assuming the points λ_i ($i = 1, 2, 3, 4$) to lie in the z -plane as indicated in Figure 1 we can for example connect any point in the interior of the overlapping sector of Figure 1 with $\infty + 0\sqrt{-1}$ by a path $\mathcal{P}$ consisting of at most two straight lines along each of which $|\arg \frac{dz}{dx}| \leq \frac{\pi}{2} - \alpha$ and neither of which passes through the origin.

In the general rank $r + 1$ (distinct eigenvalue) extreme eigenvalue case we assume $C(z)$ is holomorphic in a domain $\mathcal{A}$ extending to infinity in the direction $\arg z = 2\pi k/(r + 1)$ for some integer k in $0 \leq k \leq r$. Let $\mathcal{E}$ be the smallest closed disc with center at the origin containing the zeros of $q_{ij}'(t)$ ($i = 1, 2, \dots, n; i \neq j$), and let $\mathcal{A}^* = \mathcal{A} - \mathcal{E} - \{\infty\}$. We next define a region $\mathcal{A}(z, \zeta_k)$ to be the union of all points $z (\neq \zeta_k)$ such that there is a path $\mathcal{P}$ connecting $z \in \mathcal{A}(z, \zeta_k)$ and $\zeta_k = \infty \exp [2\pi k \sqrt{-1}/(r + 1)]$ satisfying the following extreme eigenvalue conditions:

- 1) Except for the point ζ_k , $\mathcal{P}$ lies entirely in $\mathcal{A}^*$;
- 2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $\frac{dt(s)}{ds}$ is continuous and non-vanishing;
- 3) For any two points t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathcal{P}$ we have $|\exp [q_{ij}(t_2) - q_{ij}(t_1)]| \leq 1$ ($i = 1, 2, \dots, n$);
- 4) $\gamma_{\mathcal{P}}(t^{-1}) = \int_{\mathcal{P}} |t^{-2} dt|$ is bounded.

We have already seen that for a particular rank 1 case with eigenvalues as in Figure 1 there always is such a path connecting each point z in $|\arg z| \leq \frac{3\pi}{2} - \alpha - \delta$ with $+\infty$, where δ is an arbitrary positive number less than $\frac{3}{4}\pi$. For arbitrary rank $r + 1$ it follows that, since $|\arg q_{ij}(z) - \arg [(q_{i0} - q_{j0})z^{r+1}/(r + 1)]|$ can be made arbitrarily small by taking $|z|$ sufficiently large, there is a path $\mathcal{P}$ satisfying the above conditions which connects any z ($|z|$ sufficiently large) in

$$|\arg z - \frac{2\pi k}{r + 1}| \leq \frac{1}{r + 1} \left(\frac{3\pi}{2} - \alpha - \delta \right) \text{ and } \zeta_k = \exp \left[\frac{2\pi k \sqrt{-1}}{r + 1} \right].$$

Let us set

$$(3.3.8) \quad D'_j(z) = Q'(z) - q'_j(z)I.$$

If a vector $W_j(z)$ satisfies (2.3.3), then by (3.3.2) the error vector

$$(3.3.9) \quad \epsilon_{jm}(z) = \left(W_j(z) - \varphi_{jm}(z) \right) e^{-q_j(z)}$$

satisfies

$$(3.3.10) \quad \frac{d}{dz} \epsilon_{jm}(z) - D'_j(z) \epsilon_{jm}(z) = \frac{1}{z^2} B(z) \epsilon_{jm}(z) - R_{jm}(z).$$

Now if $\epsilon_{jm;k}(z)$ is a solution of

$$(3.3.11) \quad \epsilon_{jm;k}(z) = \int_{\xi_k}^z e^{D_j(z) - D_j(t)} [t^{-2} B(t) \epsilon_{jm;k}(t) - R_{jm}(t)] dt$$

where the path of integration γ is chosen as described above, then $\epsilon_{jm;k}(z)$ simultaneously also satisfies (3.3.10).

Let us now turn to the interior eigenvalue case. We recall that we assume all eigenvalues of A_0 to lie in the shaded region indicated in Figure 2.

Take for example a rank one case for which the coefficient matrix $C(z)$ is holomorphic for all $|z|$ sufficiently large. If the eigenvalues of C_0 (or A_0) are as indicated in Figure 2 then given any number δ in $0 < \delta < \pi - \alpha$ there is a path γ joining $-\infty$, any point z in $|\arg z - \frac{1}{2}\pi| \leq \pi - \alpha - \delta$ and ∞ , and consisting of at most

three straight lines, none of which contacts the origin such that $\operatorname{Re}(\lambda_1 t)$, $\operatorname{Re}(\lambda_2 t)$ increase monotonically while $\operatorname{Re}(\lambda_3 t)$ decreases monotonically.

If in addition a domain about the origin is to be avoided by $\mathcal{O}$, $|z|$ may need to be taken large if δ is small. Similarly it is easy to see that there is such a path joining $-\infty$, any point z in $|\arg z + \frac{1}{2}\pi| \leq \pi - \alpha - \delta$ and ∞ . Thus these two families of paths, one of which passes around the origin in a positive sense and the other in a negative as we traverse $\mathcal{O}$ from $-\infty$ to ∞ , cover the complete vicinity of infinity.

In general assume $C(z)$ is holomorphic in a domain $\mathcal{D}$ which contains (or extends to infinity in) one of the sectors $|\arg z - (2k + \frac{1}{2}) \frac{\pi}{r+1}| \leq (\pi - \alpha)/(r+1)$ for some integer k in $0 \leq k \leq r$ and all z sufficiently large. Divide the integers $N = \{1, 2, \dots, n\}$ into two disjoint classes N_1 and N_2 such that if λ_i is in $|\arg \lambda| \leq \alpha$ then $i \in N_1$, while if λ_i is in $|\arg \lambda - \pi| \leq \alpha$ then $i \in N_2$. Note that j may be taken either in N_1 or in N_2 .

We next reorder and if necessary relabel the elements of the equation (3.3.10) so that we may write them in the form

$$\begin{aligned} \frac{d}{dz} \epsilon_{jm1}(z) - D'_{j1}(z) \epsilon_{jm1}(z) &= z^{-2} [\alpha(z) \epsilon_{jm1}(z) + \beta(z) \epsilon_{jm2}(z)] - R_{jm1}(z) \\ (3.3.12) \quad \frac{d}{dz} \epsilon_{jm2}(z) - D'_{j2}(z) \epsilon_{jm2}(z) &= z^{-2} [\gamma(z) \epsilon_{jm1}(z) + \delta(z) \epsilon_{jm2}(z)] - R_{jm2}(z) \end{aligned}$$

where the top line contains all rows of (3.3.10) such that $i \in N_1$ and the bottom line contains all rows of (3.3.10) such that $i \in N_2$. The diagonal matrix $D'_j(z)$ in (3.3.10) is accordingly subdivided into two

diagonal matrices $D'_{j1}(z)$ and $D'_{j2}(z)$; the matrix $B(z)$ is accordingly partitioned into four blocks $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$, and $R_{jm}(z)$ is split into two vectors in the same manner as $\epsilon_{jm}(z)$ was split.

Let $\mathcal{E}$ be the smallest closed disc with center at the origin containing the zeros of $q_{ij}^*(t)$ ($i = 1, 2, \dots, n$; $i \neq j$) and let $\mathcal{O}^* = \mathcal{O} - \mathcal{E} - \{\infty\}$. We next define a region $\mathcal{O}(\zeta_{1k}, \zeta_{2k})$ to be the union of all points z ($z \neq \zeta_{1k}$, $z \neq \zeta_{2k}$) such that there is a path connecting ζ_{1k} , z and ζ_{2k} (in that order), where ζ_{1k} is one of the points $\infty \exp[(2k+1)\pi\sqrt{-1}/(r+1)]$, and $\zeta_{2k} = \infty \exp[2k\pi\sqrt{-1}/(r+1)]$ satisfying the following interior eigenvalue conditions:

- 1) Except for ζ_{1k} , ζ_{2k} , $\mathcal{O}$ lies entirely in $\mathcal{O}^*$;
- 2) $\mathcal{O}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $\frac{dt(s)}{ds}$ is continuous and non-vanishing;
- 3) For any two distinct points t_1 and t_2 in the order ζ_{1k} , t_1 , t_2 , ζ_{2k} on $\mathcal{O}$ we have $|\exp[q_{ij}(t_2) - q_{ij}(t_1)]| \leq 1$ if $i \in N_1$, $|\exp[q_{ij}(t_1) - q_{ij}(t_2)]| \leq 1$ if $i \in N_2$;
- 4) With α , β , γ and δ defined by equation (3.3.30), $\frac{1}{2} \{\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\} \gamma_{\mathcal{O}}(t^{-1}) < 1$.

It follows that the vector of analytic functions which satisfies

$$(3.3.13) \quad \begin{bmatrix} \epsilon_{jm1;k}(z) \\ \epsilon_{jm2;k}(z) \end{bmatrix} = \begin{bmatrix} \int_{\zeta_{1k}}^z e^{D_{j1}(z)-D_{j1}(t)} \\ \int_{\zeta_{2k}}^z e^{D_{j1}(z)-D_{j2}(t)} \end{bmatrix} \left\{ t^{-2} \begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix} \right. \\ \left. \begin{bmatrix} \epsilon_{jm1;k}(t) \\ \epsilon_{jm2;k}(t) \end{bmatrix} - \begin{bmatrix} R_{jm1}(t) \\ R_{jm2}(t) \end{bmatrix} \right\} dt$$

where the integrals are taken along a path $\mathcal{P}$ as described above, is simultaneously a solution of (3.3.12).

3.3.3 Treatment of $R_{ijm}(z)$

In this section we shall accurately estimate the integral

$$(3.3.14) \quad R_{ijm}^*(z) = \int_{\zeta}^z e^{q_{ij}(z)-q_{ij}(t)} R_{ijm}(t) dt$$

where the path $\mathcal{P}$ of integration satisfies anyone of the sets of conditions described in the previous section and ζ is a point appropriately chosen at infinity such that as t traverses $\mathcal{P}$ from ζ to z , $\operatorname{Re} q_{ij}(t)$ decreases monotonically. $R_{ijm}(t)$ is given by (3.3.7).

The chief difficulty surrounding the above integral occurs for $i \neq j$. If $i = j$ then $R_{ijm}(t) = O(t^{-m-1})$, $|t| \rightarrow \infty$; if $i \neq j$ then $R_{ijm}(t) = O(t^{r-m-1})$, $|t| \rightarrow \infty$. Thus when $i \neq j$ we integrate

(3.3.14) by parts to obtain a good bound on $R_{ijm}(z)$.

For $i \neq j$, the equation (3.3.7) may be written

(3.3.15)

$$R_{ijm}(z) = -e^{q_{ij}(z)} \frac{d}{dz} (e^{-q_{ij}(z)} U_{ijm} z^{-m}) - z[q'_{ij}(z) - (q_{io} - q_{jo})z^r + \frac{m}{z}] U_{ijm} z^{-m-1} \\ - \sum_{\mu=m+1}^{m+r+1} \left[\sum_{k=\mu+1-m}^{\min(\mu, r+1)} (q_{ik} - q_{jk}) U_{ij, \mu-k} + (\mu-r-1) U_{ij, \mu-r-1} + \sum_{k=0}^{\mu-r-2} \{B_k U_{j, \mu-r-2-k}\}_i \right] z^{r-\mu} \\ - \sum_{k=0}^{m-1} \left\{ \left(B_{m-k}(z) U_{jk} \right) \right\}_i z^{-m-2}.$$

Thus the first term on the right of (3.3.15) has a form suitable for bounding. Consider now that polynomial part $P_{ijm}(z)$ consisting of all of the right hand side of (3.3.15) excluding the first and last terms. It is clear that $P_{ijm}(z)$ is a polynomial with dominating term of order z^{r-m-1} as $|z| \rightarrow \infty$. On substituting $P_{ijm}(t)$ for $R_{ijm}(t)$ in (3.3.14) and denoting the resulting integral by $P_{ijm}^*(z)$ we have

$$(3.3.16) \quad P_{ijm}^*(z) = \int_{\zeta}^z e^{q_{ij}(z) - q_{ij}(t)} q'_{ij}(t) \left(\frac{1}{q'_{ij}(t)} P_{ijm}(t) \right) dt$$

so that

$$(3.3.17) \quad P_{ijm}^*(z) = - \frac{P_{ijm}(z)}{q'_{ij}(z)} + \int_{\zeta}^z e^{q_{ij}(z) - q_{ij}(t)} \frac{d}{dt} \left(\frac{P_{ijm}(t)}{q'_{ij}(t)} \right) dt.$$

Collecting terms we obtain

$$(3.3.18) \quad R_{ijm}^*(z) = \int_{\xi}^z \left[m U_{ijm} t^{-m-1} + \left(e^{q_{ij}(z)-q_{ij}(t)} - 1 \right) \frac{d}{dt} \left(\frac{P_{ijm}(t)}{q'_{ij}(t)} \right) \right. \\ \left. - e^{q_{ij}(z)-q_{ij}(t)} \sum_{k=0}^{m-1} \left\{ B_{m-k}(t) U_{jk} \right\}_i t^{-m-2} \right] dt ,$$

and since $|e^{q_{ij}(z)-q_{ij}(t)}| \leq 1$, and $\left| \frac{d}{dt} \left(\frac{P_{ijm}(t)}{q'_{ij}(t)} \right) \right| \leq O(|t|^{-m-2})$

$|t| \rightarrow \infty$, the right of (3.3.18) is easily bounded and can be utilized for purposes of either a vector or a norm bound.

On the other hand, for $i = j$ we have by (3.3.5) that

$$(3.3.19) \quad R_{jjm}^*(z) = \int_{\xi}^z \left[m U_{jjm} t^{-m-1} - \sum_{k=0}^{m-1} \left\{ B_{m-k}(z) U_{jk} \right\}_j t^{-m-2} \right] dt$$

and obtaining a bound for this equation is even simpler than obtaining one for (3.3.18).

3.3.4 Error Bounds for the Extreme Eigenvalue Case

We shall first obtain a vector bound. By Section B.2 of Appendix B the solution of the equation (3.3.11) exists and is a unique vector of functions holomorphic in a Riemann surface which contains the region $\mathcal{R}(z, \zeta_k)$ defined by the extreme eigenvalue conditions in Section 3.3.2.

With t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathcal{P}$ we define

$$(3.3.20) \quad \begin{aligned} (m+1) \gamma_{ijm} = & \sup_{t_1, t_2 \in \mathcal{P}} \left| e^{q_{ij}(t_2) - q_{ij}(t_1)} \left[t_1^{m+2} \frac{d}{dt} \left(\frac{p_{ijm}(t)}{q'_{ij}(t)} \right) \right]_{t=t_1} + \right. \\ & \left. + \sum_{k=1}^{m-1} \left\{ B_{m-k}(t_1(U_{jk})_i) \right\} - \frac{d}{dt} \left(\frac{p_{ijm}(t)}{q'_{ij}(t)} \right) \right]_{t=t_1} t_1^{m+2} \right| \end{aligned}$$

$$i = 1, 2, \dots, n; i \neq j)$$

$$(m+1) \gamma_{jjm} = \sup_{t \in \mathcal{P}} \left| \sum_{k=0}^{m-1} \left\{ B_{m-k}(t) U_{jk} \right\}_j \right|.$$

We also denote γ_{jm} to be the vector with i 'th element γ_{ijm} ; i.e.,

$$\{\gamma_{jm}\}_i = \gamma_{ijm}, \text{ and}$$

$$(3.3.21) \quad B = \left[\sup_{t_1, t_2 \in \mathcal{P}} \left| e^{q_{ij}(t_2) - q_{ij}(t_1)} \left\{ B(t_1) \right\}_{is} \right| \right] \quad (i, s = 1, 2, \dots, n).$$

On combining equations (3.3.20) with (3.3.21) and substituting in equation (3.3.11) we obtain

$$(3.3.22) \quad |\epsilon_{jm;k}(z)| \leq \int_{\zeta_k}^z \left[|t^{-2}| B |\epsilon_{jm;k}(t)| + m |U_{jm} t^{-m-1}| + (m+1) \gamma_{jm} |t^{-m-2}| \right] |dt|.$$

We apply Lemmas C.1 and C.2 of Appendix C to (3.3.22) to obtain

Theorem 3.3.1 If corresponding to an extreme eigenvalue of A_0
(or C_0) we can define a region $\mathcal{O}(z, \zeta_k)$ as described by the extreme
eigenvalue conditions in Section 3.3.2, then the equation (2.3.3) possesses

an actual solution vector $W_{jm;k}(z)$ depending on ζ_k and on arbitrary
positive integer m such that

$$(3.3.23) \quad W_{jm;k}(z) = \left[\sum_{s=0}^{m-1} U_{js} z^{-s} + \epsilon_{jm;k}(z) \right] e^{q_j(z)}$$

where

$$(3.3.24) \quad |\epsilon_{jm;k}(z)| \leq \exp \{B \gamma_{\rho}(t^{-1})\} \{ \gamma_{\rho}(U_{jm} t^{-m}) + \gamma_{\rho}(\gamma_{jm} t^{-m-1}) \}$$

whenever z is in $\mathcal{D}(z, \zeta_k)$. Moreover $W_{jm;k}(z)$ is a vector of functions
holomorphic on a Riemann surface which contains $\mathcal{D}(z, \zeta_k)$. In (3.3.23)
each vector U_{jk} ($k = 0, 1, \dots$) is the j 'th column vector of the matrix
 U_k defined by Theorem 2.3.2; $q_j(z)$ is the j 'th diagonal element of the
diagonal matrix $Q(z)$ (equation (2.3.14)). In (2.3.24) B is an $n \times n$
matrix of non-negative elements defined through equations (2.3.4) (with
 $B_0(z) = B(z)$) and (3.3.21), while the i 'th element ($i = 1, 2, \dots, n$) of
the vector γ_{jm} is defined by equation (3.3.20).

Let us now obtain a norm bound for $\epsilon_{jm;k}(z)$ in equation
(3.3.23). For this purpose we define a vector $V_j(z, t)$ by

$$(3.3.25) \quad \{V_j(z, t)\}_i = - (e^{q_{ij}(z) - q_{ij}(t)} - 1) t^{m+2} \frac{d}{dt} \left(\frac{p_{ijm}(t)}{q'_{ij}(t)} \right) +$$

$$+ e^{q_{ij}(z) - q_{ij}(t)} \sum_{k=0}^{m-1} \{B_{m-k}(t) U_{jk}\}_i$$

$$(i = 1, 2, \dots, n; i \neq j)$$

$$= \sum_{k=0}^{m-1} \{B_{m-k}(t) U_{jk}\}_j \quad (i = j)$$

Using any one of the definitions of norms in Chapter I, and with t_1 and t_2 defined as for equation (3.3.20) we define

$$B = \sup_{t_1, t_2 \in \mathcal{D}} \|e^{D_j(t_2) - D_j(t_1)} B(t)\|$$

(3.3.26)

$$\gamma_{jm} = \sup_{t_1, t_2 \in \mathcal{D}} \|V_j(t_2, t_1)\|.$$

Theorem 3.3.2 A norm bound for the vector $e_{jm;k}(z)$ in Theorem
3.3.1 is given by

$$(3.3.27) \quad \|e_{jm;k}(z)\| \leq \exp \{B\gamma_{\mathcal{D}}(t^{-1})\} \{ \|u_{jm}\| \gamma_{\mathcal{D}}(t^{-m}) + \gamma_{jm} \gamma_{\mathcal{D}}(t^{-m-1}) \}$$

where B and γ_{jm} are non-negative numbers defined by equations (3.3.25) and (3.3.26).

3.3.5 Error Bounds for the Interior Eigenvalue Case

By the interior eigenvalue conditions of Section 3.3.2 the solution to equation (3.3.13) exists and is a unique vector of functions holomorphic in a domain (actually a Riemann surface) which contains the region $\mathcal{D}(\xi_{1k}, \xi_{2k})$. The proof of this last statement is given in Section B.2 of Appendix B.

Using any one of the definitions of norm given in Chapter I, we shall start by obtaining a norm bound for the solution of equation (3.3.13).

Let $V_j(z, t)$ be defined as in (3.3.25). We split the vector $V_j(z, t)$ into two vectors $V_{j1}(z, t)$ and $V_{j2}(z, t)$ in a manner corresponding to the way in which $R_{jm}(t)$ was split for equation (3.3.13). We analogously split the vector U_{jm} into U_{jm1} and U_{jm2} , such that equation (3.3.13) may be written

$$(3.3.28) \quad \begin{bmatrix} \epsilon_{jm1;k}(z) \\ \epsilon_{jm2;k}(z) \end{bmatrix} = \begin{bmatrix} \int_{\xi_{1k}}^z & 0 \\ 0 & \int_{\xi_{2k}}^z \end{bmatrix} \cdot \left\{ \begin{bmatrix} e^{D_{j1}(z)-D_{j1}(t)} & 0 \\ 0 & e^{D_{j2}(z)-D_{j2}(t)} \end{bmatrix} \right\} t^{-2} \cdot$$

$$\begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix} \begin{bmatrix} \epsilon_{jm1,k}(t) \\ \epsilon_{jm2,k}(t) \end{bmatrix} + \left\{ \begin{bmatrix} -m U_{jm1} t^{-m-1} + V_{j1}(z, t) t^{-m-2} \\ -m U_{jm2} t^{-m-1} + V_{j2}(z, t) t^{-m-2} \end{bmatrix} \right\} dt$$

We furthermore define

$$(3.3.29) \quad \psi_{m1}(\xi_{1k}, z) = \|U_{jm1}\| \bigvee_{\xi_{1k}, z} (t^{-m}) + \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{1}{m+1} V_{j1}(t_2, t_1) \right\| \bigvee_{\xi_{1k}, z} (t^{-m-1})$$

$$\psi_{m2}(z, \xi_{2k}) = \|U_{jm2}\| \bigvee_{z, \xi_{2k}} (t^{-m}) + \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{1}{m+1} V_{j2}(t_1, t_2) \right\| \bigvee_{z, \xi_{2k}} (t^{-m-1})$$

where t_1 and t_2 are any two points on $\mathcal{P}$ in the order $\xi_{1k}, t_1, t_2, \xi_{2k}$ and the variations are taken along $\mathcal{P}$. Similarly we put

$$\alpha = \sup_{t_1, t_2 \in \mathcal{D}} \|e^{D_{j1}(t_2) - D_{j1}(t_1)} \alpha(t_1)\|, \quad \beta = \sup_{t_1, t_2 \in \mathcal{D}} \|e^{D_{j1}(t_2) - D_{j1}(t_1)} \beta(t_1)\|$$

(3.3.30)

$$\gamma = \sup_{t_1, t_2 \in \mathcal{D}} \|e^{D_{j2}(t_1) - D_{j2}(t_2)} \gamma(t_2)\|, \quad \delta = \sup_{t_1, t_2 \in \mathcal{D}} \|e^{D_{j2}(t_1) - D_{j2}(t_2)} \delta(t_2)\|$$

Applying the analysis of Section B.2 to the above equations we obtain the following theorem.

Theorem 3.3.3 If corresponding to an interior eigenvalue λ_j of A_0 (or C_0) the interior eigenvalue conditions of Section 3.3.2 are satisfied, then the equation (2.3.3) possesses an actual solution vector

$$(3.3.31) \quad \begin{bmatrix} W_{jm1;k}(z) \\ W_{jm2;k}(z) \end{bmatrix} = \sum_{s=0}^{m-1} \begin{bmatrix} U_{js1} \\ U_{js2} \end{bmatrix} z^{-s} + \begin{bmatrix} \epsilon_{jm1;k}(z) \\ \epsilon_{jm2;k}(z) \end{bmatrix} e^{q_j(z)},$$

at each point z of $\mathcal{M}(\zeta_{1k}, \zeta_{2k})$, where

(3.3.32)

$$\begin{bmatrix} \|\epsilon_{jm1;k}(z)\| \\ \|\epsilon_{jm2;k}(z)\| \end{bmatrix} \leq \begin{bmatrix} \psi_{m1}(\zeta_{1k}, z) \\ \psi_{m2}(z, \zeta_{2k}) \end{bmatrix} + \begin{bmatrix} \sqrt{\zeta_{1k}, z}(t^{-1}) & 0 \\ 0 & \sqrt{z, \zeta_{2k}}(t^{-1}) \end{bmatrix} E f[E \sqrt{\mathcal{D}}(t^{-1})] \begin{bmatrix} \psi_{m1}(\zeta_{1k}, \zeta_{2k}) \\ \psi_{m2}(\zeta_{1k}, \zeta_{2k}) \end{bmatrix}$$

and where

$$(3.3.33) \quad E f(E_\chi) = \frac{1}{1 - (\alpha + \delta)\chi + \Delta\chi^2} \begin{bmatrix} \alpha - \Delta\chi & \beta \\ \gamma & \delta - \Delta\chi \end{bmatrix}; \quad \Delta = \alpha\delta - \beta\gamma.$$

The bound (3.3.32) is valid for all z such that $\frac{1}{2} \{ \alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} \chi(t^{-1}) < 1$.

The elements of the solution vector (3.3.31) depend on ζ_{1k}, ζ_{2k} and an arbitrary positive integer m ; each element in this vector is holomorphic in a Riemann surface which contains $\mathcal{Q}(\zeta_{1k}, \zeta_{2k})$. The function $q_j(z)$ is the j 'th diagonal element of the diagonal matrix $Q(z)$ (equation (2.3.14)).

The vector U_{js} , the j 'th column vector of U_s ($s = 0, 1, 2, \dots$) (defined in the proof of Theorem 2.3.2) is partitioned in a manner described in

Section 3.3.2 into two lower dimensional vectors U_{js1} and U_{js2} .

The functions $\psi_{ms}(\cdot, \cdot)$ ($s = 1, 2$) are defined by equation (3.3.29) and the numbers α, β, γ and δ are defined by equation (3.3.30).

Next, we shall obtain a vector bound for the solution of equation (3.3.13). To achieve this goal we proceed directly to bounding the elements of vectors and matrices in equation (3.3.28). Thus, with

$$(3.3.34) \quad \alpha(t) = [\{\alpha(t)\}_{is} (i \in N_1, s \in N_1)], \quad \beta(t) = [\{\beta(t)\}_{is} (i \in N_1, s \in N_2)]$$

$$\gamma(t) = [\{\gamma(t)\}_{is} (i \in N_2, s \in N_1)], \quad \delta(t) = [\{\delta(t)\}_{is} (i \in N_2, s \in N_2)]$$

and t_1 and t_2 as defined for equation (3.3.29) we define

$$\alpha = \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ e^{D_{j1}(t_2) - D_{j1}(t_1)} \alpha(t_1) \}_{is} | \right], \beta = \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ e^{D_{j1}(t_2) - D_{j1}(t_1)} \beta(t_1) \}_{is} | \right]$$

(3.3.36)

$$\gamma = \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ e^{D_{j2}(t_1) - D_{j1}(t_2)} \gamma(t_2) \}_{is} | \right], \delta = \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ e^{D_{j2}(t_1) - D_{j2}(t_2)} \delta(t_2) \}_{is} | \right]$$

and

$$C_{m1}(t) = m |U_{jm1}| |t^{-m-1}| + \sup_{t_1, t_2 \in \mathcal{P}} |V_{j1}(t_2, t_1)| |t^{-m-2}|$$

(3.3.37)

$$C_{m2}(t) = m |U_{jm2}| |t^{-m-1}| + \sup_{t_1, t_2 \in \mathcal{P}} |V_{j2}(t_1, t_2)| |t^{-m-2}|$$

where t_1 and t_2 are points on $\mathcal{P}$ located as described for equation

(3.3.29). In addition, each element of the diagonal matrices

$e^{D_{js}(z) - D_{js}(t)}$ ($s = 1, 2$) has an upper bound of 1. On combining the

above results we obtain from (3.3.28)

$$(3.3.38) \quad \begin{bmatrix} |\epsilon_{jm1,k}(z)| \\ |\epsilon_{jm2,k}(z)| \end{bmatrix} \leq \begin{bmatrix} \int_{\xi_{1k}}^z & 0 \\ 0 & \int_{\xi_{2k}}^z \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} |\epsilon_{jm1,k}(t)| \\ |\epsilon_{jm2,k}(t)| \end{bmatrix} |t^{-2}| + \begin{bmatrix} C_{m1}(t) \\ C_{m2}(t) \end{bmatrix} \Bigg\} |dt|$$

The inequality (3.3.38) is solved in Section C.3 of Appendix C.

Collecting the results pertaining to the solution of this inequality we

have

Theorem 3.3.4 If all eigenvalues of the matrix $C_2 C_1 (\zeta_{1k}, \zeta_{2k})$ (see equation (3.3.40)) are less than one in magnitude then the vector $\epsilon_{jm1;k}(z)$ (equation (3.3.31)) is also bounded as

$$(3.3.39) \quad |\epsilon_{jm1;k}(z)| \leq \exp \{ \alpha \gamma_{\zeta_{1k}, z} (\xi^{-1}) \} \psi_{m1}(\zeta_{1k}, z) \\ + C_1(\zeta_{1k}, z) [\exp \{ \delta \gamma_{\varphi} (\xi^{-1}) \} \psi_{m2}(\zeta_{1k}, \zeta_{2k}) + B],$$

where

$$\psi_{ms}(u, v) = \int_u^v C_{ms}(t) |dt| \quad (s = 1, 2)$$

$$B = [I_2 - C_2 C_1(\zeta_{1k}, \zeta_{2k})]^{-1} C_2.$$

$$(3.3.40) \quad [\exp \{ \alpha \gamma_{\varphi} (\xi^{-1}) \} \psi_{m1}(\zeta_{1k}, \zeta_{2k}) + C_1(\zeta_{1k}, \zeta_{2k}) \exp \{ \delta \gamma_{\varphi} (\xi^{-1}) \} \psi_{m2}(\zeta_{1k}, \zeta_{2k})]$$

$$C_1(\zeta_{1k}, z) = \alpha^{-1} [\exp \{ \alpha \gamma_{\zeta_{1k}, z} (\xi^{-1}) \} - I_1] \beta; \quad C_2 = \delta^{-1} [\exp \{ \delta \gamma_{\varphi} (\xi^{-1}) \} - I_2]$$

and where I_1 designates a $\kappa \times \kappa$ unit matrix, and I_2 designates an $(n - \kappa) \times (n - \kappa)$ unit matrix; κ being the number of elements in N_1 (Section 3.3.2). Moreover, an exactly similar result holds for $\epsilon_{jm2;k}(z)$.

A norm bound which is sharper in the neighborhood of infinity than that given in Theorem 3.4.3 can be obtained by a procedure similar to the one used to obtain the vector bound in Theorem 3.4.4 (see Appendix C).

Theorem 3.3.5 With the notation of Theorem 3.3.3 let $\Omega = \max(\alpha, \delta)$. If

$$(3.3.41) \quad C = \frac{\beta\gamma}{\Omega(\alpha+\delta)} \exp \left\{ \left(\Omega + \frac{|\alpha-\delta|}{2} \right) \mathcal{V}_{\varphi}(\xi^{-1}) \right\} \sinh \left\{ \frac{\alpha+\delta}{2} \mathcal{V}_{\varphi}(\xi^{-1}) \right\} < 1$$

then the norm of the solution $\epsilon_{j_{m1},k}(z)$ (equation (3.3.31)) is bounded as

$$(3.3.42) \quad \begin{aligned} \|\epsilon_{j_{m1},k}(z)\| \leq & \exp \left\{ \alpha \mathcal{V}_{\xi_{1k},z}(\xi^{-1}) \right\} \psi_{m1}(\xi_{1k}, z) \\ & + Y(z) \{ \psi_{m2}(\xi_{1k}, \xi_{2k}) + \gamma(1-C)^{-1} [D(\alpha) \psi_{m1}(\xi_{1k}, \xi_{2k}) + E \psi_{m2}(\xi_{1k}, \xi_{2k})] \} \end{aligned}$$

where

$$Y(z) = \frac{2\beta}{\alpha+\delta} \exp \left\{ \left(\Omega + \frac{|\alpha-\delta|}{2} \mathcal{V}_{\varphi}(\xi^{-1}) \right) \sinh \left\{ \frac{\alpha+\delta}{2} \mathcal{V}_{\xi_{1k},z}(\xi^{-1}) \right\} \right\}$$

$$(3.3.43) \quad D(\chi) = \frac{1}{\chi} [\exp \{ \chi \mathcal{V}_{\varphi}(\xi^{-1}) \} - 1]$$

$$E = \frac{\beta}{\alpha+\delta} [D(\alpha+\delta) - 2 \mathcal{V}_{\varphi}(\xi^{-1}) + D(\alpha) \exp \{ \delta \mathcal{V}_{\varphi}(\xi^{-1}) \}].$$

It is noteworthy that the conditions of Theorem 3.3.4 can be tested using the Gersgoren theorems ([12]p. 227) and that they (as well as the conditions of Theorems 3.3.3 and 3.3.5) can be satisfied by restricting ourselves to contours sufficiently far from the origin.

3.4 The General Case

3.4.1 The Differential Equation for an Approximation

We shall obtain an actual solution vector of the system (2.4.4)-(2.4.5) corresponding to the v 'th formal solution vector (as defined in equation (2.4.22)) in the j 'th column of blocks of formal solutions as defined in the proof of Theorem 2.4.2.

Starting with the partial sum

$$(3.4.1) \quad \varphi_{jm}(z) = \left(\sum_{k=0}^{m-1} U_{jk} z^{-k} \right) e^{Q_j(z)}$$

where U_{jk} is the j 'th column of blocks of the matrix U_k ($k = 0, 1, \dots$) as defined in the proof of Theorem 2.4.2, and

$$(3.4.2) \quad Q_j(z) = q_j(z) I_{\mu_j} + J_{\mu_j} \log z,$$

$q_j(z)$, I_{μ_j} and J_{μ_j} also being defined in Theorem 2.4.2, we define $R_{jm}(z)$ by the differential equation

$$(3.4.3) \quad \frac{d}{dz} \varphi_{jm}(z) - [Q'(z) + z^{-2} B(z)] \varphi_{jm}(z) = R_{jm}(z) e^{Q_j(z)}.$$

On expanding (3.4.3) and using (2.4.10) we obtain

$$(3.4.4) \quad R_{jm}(z) = - \sum_{\mu=0}^{r+m+1} \left\{ \sum_{s=0}^{\min(\mu, r+1)} (Q_s U_{j, \mu-s}^+ - U_{j, \mu-s}^+ Q_{js}) \right. \\ \left. + (\mu-r-1) U_{j, \mu-r-1}^+ + \sum_{s=0}^{\mu-r-2} B_s U_{j, \mu-r-2-s} \right\} z^{r-\mu} - \sum_{s=0}^{m-1} \left(B_{m-s}(z) U_{jk} \right) z^{-m-2}$$

where $U_{js}^+ = \{U_{js} \text{ if } 0 \leq s \leq m-1; 0 \text{ otherwise}\}$, $Q_s = (s=0, 1, \dots, r+1)$ is defined in (2.4.7), Q_{js} is the j 'th block of Q_s , and $B_s(z)$ ($s = 1, 2, \dots$) is defined in (2.4.8).

It is necessary to examine the (i, j) 'th block $R_{ijm}(z)$ of equation (3.4.4). On making use of (2.4.21) in the (i, j) 'th block of (3.4.4) we obtain

$$\begin{aligned}
 (3.4.5) \quad R_{ijm}(z) &= q'_{ij}(z)U_{ijm}z^{-m} - [q'_{ij}(z) - (q_{ip} - q_{jp})z^{r-p}]U_{ijm}z^{-m} \\
 &- \sum_{\mu=m+p+1}^{m+r+1} \left[\sum_{s=\mu+1-m}^{\min(\mu, r+1)} (q_{is} - q_{js})U_{ij, \mu-s} + J_{\mu_i} U_{ij, \mu-r-1} - U_{ij, \mu-r-1} J_{\mu_j} \right. \\
 &+ (\mu-r-1)U_{ij, \mu-r-1} + \left. \sum_{s=0}^{\mu-r-2} \left\{ B_s U_{j, \mu-r-2-s} \right\}_i \right] z^{r-\mu} \\
 &- \sum_{s=0}^{m-1} \left\{ B_{m-s}(z) U_{js} \right\}_i z^{-m-2}
 \end{aligned}$$

if $i \neq j$ and for some p in $0 \leq p \leq r$, $q_{ip} - q_{jp} \neq 0$. On the other hand if $q_{ip} = q_{jp}$ ($0 \leq p \leq r$), then

$$\begin{aligned}
 (3.4.6) \quad R_{ijm}(z) &= [(q_{i, r+1} - q_{j, r+1} + m)U_{ijm} - U_{ijm} J_{\mu_j} + J_{\mu_i} U_{ijm}] z^{-m-1} \\
 &- \sum_{s=0}^{m-1} \left\{ B_{m-s}(z) U_{js} \right\}_i z^{-m-2}
 \end{aligned}$$

where (3.4.6) also includes the case $i = j$.

Let a block $W_j(z)$ of μ_j vectors of holomorphic functions satisfy the differential equation (2.4.4)-(2.4.5). The error block

$$(3.4.7) \quad \eta_{jm}(z) = W_j(z) - \Phi_{jm}(z)$$

satisfies the equation

$$(3.4.8) \quad \frac{d}{dz} \eta_{jm}(z) - [Q(z) + z^{-2} B(z)] \eta_{jm}(z) = - R_{jm}(z) e^{Q_j(z)}$$

where $R_{jm}(z)$ is given by (3.4.5) or (3.4.6) above.

For later convenience we partition the equation (3.4.8) into two simultaneous blocks

$$(3.4.9) \quad \frac{d}{dz} \begin{bmatrix} \eta_{jm1}(z) \\ \eta_{jm2}(z) \end{bmatrix} = \begin{bmatrix} D'_{j1}(z) + I_1 Q'_j(z) & 0 \\ 0 & D'_{j2}(z) + I_2 Q'_j(z) \end{bmatrix} + z^{-2} \begin{bmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix} \begin{bmatrix} \eta_{jm1}(z) \\ \eta_{jm2}(z) \end{bmatrix} = - \begin{bmatrix} R_{jm1}(z) \\ R_{jm2}(z) \end{bmatrix} e^{Q_j(z)}$$

where $\eta_{jm1}(z)$ contains all those i 'th blocks for which $i \in N_1$; $\eta_{jm2}(z)$ contains all those i 'th blocks for which $i \in N_2$ where N_1 and N_2 are two disjoint sets with $N_1 \cup N_2 = \{1, 2, \dots, \ell\}$. Precise definitions of N_1 and N_2 will be given in the following section. The matrix $B(z)$ is

accordingly partitioned into blocks $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$, and $R_{jm}(z)$ is similarly partitioned into $R_{jm1}(z)$ and $R_{jm2}(z)$. The matrices $D_{j1}(z)$ and $D_{j2}(z)$ are partitions of the matrix $D_j(z) = Q(z) - Iq_j(z)$.

3.4.2 The Different Possible Cases

Some of the possible cases that can arise in the most general case have already been covered; namely the case of a regular point, the case of a regular singular point and the case of an irregular singular point when the eigenvalues of the lead coefficient matrix are distinct. In addition we have assumed in all of the above cases that the end points of integration in the integral equations defining solutions can always be chosen at infinity. By this last assumption we have merely assumed that the coefficient matrix is holomorphic in certain domains extending to infinity. However, when two or more eigenvalues of the lead coefficient matrix are the same, or more generally for $C(z)$ as described in Theorem 2.4.1 it will not always be possible to achieve end-points of paths at infinity, even when $C(z)$ is holomorphic for all $|z|$ sufficiently large.

In our treatment of the general case we shall cover all the possible remaining cases, including those already covered in previous sections.

It is demonstrated in [8] that for each fixed j the complete vicinity of infinity can be divided into a finite number of sectors $S_{k,j}$ ($k = 1, 2, \dots, N$) such that for all i ($i = 1, 2, \dots, \ell$) for which there is an integer p in $0 \leq p \leq r$ such that $q_{ip} - q_{jp} \neq 0$ we have either $\operatorname{Re} q_{ij}(z)$ bounded above or else $\operatorname{Re} q_{ij}(z)$ tending to $+\infty$ as $|z| \rightarrow \infty$

in a particular sector S_{kj} . This fact enables us to classify all the remaining cases under two different headings: (a) those cases for which it is possible to define an actual solution vector by a single Volterra integral equation; and (b) those cases for which we require a simultaneous pair of Volterra integral equations to define a solution vector.

In order to characterize cases (a) and (b) above, we first establish our notation. $\{\zeta\} \in \bar{\mathcal{D}}$ ($\mathcal{D}$ is the domain in which $c(z)$ (equation (2.4.4)) is holomorphic) will denote the fixed limits of integration: in case (a), $\{\zeta\} = \zeta$; in case (b), $\{\zeta\} = \zeta_1, \zeta_2$. $\mathcal{E}_a$ will denote a closed disc with center at the origin and radius a . If for all i ($i = 1, 2, \dots, \ell$) $q_{ip} = q_{jp}$, $p = 0, 1, \dots, r$, then $a = 0$ if every member of $\{\zeta\}$ is unbounded, while if $\min \{|\zeta|\} < \infty$ we set $a = \min \{|\zeta|\}$. If in case (b) all members of $\{\zeta\}$ are bounded then we assume $\zeta_1 \neq \zeta_2$ and $a = |\zeta_1| = |\zeta_2|$. If for at least one $i \in (1, 2, \dots, \ell)$ we have $q_{ip} \neq q_{jp}$ for some $p \in (0, 1, \dots, r)$, let a_0 denote the maximum modulus of the set of zeros of $q'_{ij}(z)$ for all such i , set $a = a_0$ if $\min \{|\zeta|\} = \infty$ while if $\min \{|\zeta|\} < \infty$ we choose $\{\zeta\}$ such that $\min \{|\zeta|\} \geq a_0$ and no member of $\{\zeta\}$ coincides with a zero of $q'_{ij}(z)$. Let $\mathcal{D}^* = \mathcal{D} - \mathcal{E}_a - S$ where S is empty or $\{\infty e^{i\theta} \mid -\infty < \theta < \infty\}$ according to whether or not $r \leq -2$.

Case (a) above is related to the extreme eigenvalue case of the previous section. It is characterized as follows. We define a region $\mathcal{D}(z, \zeta)$ to be the union of all points z ($\neq \zeta$) such that there is a path $\mathcal{P}$ connecting $z \in \mathcal{D}(z, \zeta)$ and ζ which satisfies the following conditions:

- 1) Except for ζ , if $\zeta \in \bar{\mathcal{D}}^* - \mathcal{D}^*$, $\mathcal{P}$ lies entirely in $\mathcal{D}^*$;

2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $\frac{dt(s)}{ds}$ is continuous and non-vanishing;

3) All elements of the matrices $\exp \{Q_i(\tau) - Q_i(t) - I_{\mu_i} [q_j(\tau) - q_j(t)]\}$ ($i = 1, 2, \dots, l; i \neq j$) are bounded for all points t, τ in the order ζ_k, t, τ, z on $\mathcal{P}$. If $J_{\mu_j} \neq 0$, or if for some i ($i = 1, 2, \dots, l; i \neq j$) we have $q_{ip} = q_{jp}$ for $p = 0, 1, \dots, r$ (or both) we also require that τ/t be bounded;

4) $\gamma_{\mathcal{P}}(t^{\eta-1})$ is bounded for each fixed η where η may be taken zero if $J_{\mu_j} = 0$ but otherwise η is an arbitrary positive number less than 1.

It is shown in Appendix B that if $\mathcal{O}(z, \zeta)$ contains a point z_0 (z_0 bounded, $\neq \zeta$) then $\mathcal{O}(z, \zeta)$ is in fact a non-empty open connected domain.

We observe here that the region $\mathcal{O}(z, \zeta)$ defined by the above conditions includes the corresponding region defined in Section 3.1 and 3.2.2 in the case when the general system reduces to that at a regular point, or a regular singular point, and it similarly includes the region defined by the extreme eigenvalue condition in Section 3.3.2 in the extreme eigenvalue case when the lead coefficient matrix of the general system has all its eigenvalues distinct.

Case (b) above is related to the interior eigenvalue case of the previous section. It is characterized as follows. We define a region $\mathcal{O}(\zeta_1, \zeta_2, \eta)$ such that there is a path $\mathcal{P}$ connecting ζ_1, z and ζ_2 (in that order) satisfying the following conditions:

1) Except for ζ_1 and ζ_2 which may be boundary points of $\mathcal{O}^*$, $\mathcal{P}$ lies entirely in $\mathcal{O}^*$;

2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $\frac{dt(s)}{ds}$ is continuous and non-vanishing;

3) It is possible to subdivide the integers $i = 1, 2, \dots, \ell$ into two non-empty disjoint classes N_1 and N_2 such that for all points t and τ in the order $\zeta_1, t, \tau, \zeta_2$ on $\mathcal{P}$, all elements of the matrix $Q_{ij}(\tau, t) = \exp \{Q_i(\tau) - Q_i(t) - I_{\mu_i} [q_j(\tau) - q_j(t)]\}$ are bounded if $i \in N_1$ ($i \neq j$) while if $i \in N_2$ ($i \neq j$) then all elements of the matrix $Q_{ij}(t, \tau)$ are bounded. If $J_{\mu_j} \neq 0$, or if for some i ($i = 1, 2, \dots, \ell; i \neq j$) we have $q_{ip} = q_{jp}$ for $p = 0, 1, \dots, r$ or both then we assume without loss of generality that both i and j are in N_2 , that ζ_1 is bounded and that t/τ remains bounded for all points t, τ in the order $\zeta_1, t, \tau, \zeta_2$ on $\mathcal{P}$;

4) $\frac{1}{2} \{ \alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} \} \gamma_{\mathcal{P}}(t^{\eta-1}) < 1$, where η may be taken to be zero if $J_{\mu_j} = 0$, but otherwise η is an arbitrary positive number less than 1, and α, β, γ and δ are defined by (3.4.31).

It is clear that in the case when the eigenvalues of the lead coefficient matrix of the general system are distinct then the domain

$\mathcal{D}(\zeta_1, \zeta_2, 0)$ defined by the above conditions includes the domain $\mathcal{D}(\zeta_{1k}, \zeta_{2k})$ defined by the interior eigenvalue conditions in Section 3.3.2.

We have shown in Appendix B that if $\mathcal{D}(\zeta_1, \zeta_2, \eta)$ contains a point z_0 ($|z_0|$ bounded) distinct from ζ_1 and ζ_2 then $\mathcal{D}(\zeta_1, \zeta_2, \eta)$ is in fact an open connected domain.

If $r \geq 0$ (equation (2.4.4)) we can again represent the eigenvalues of C_0 as points in the complex plane and there enclose them by a smallest strictly convex closed polygon. As opposed to the distinct eigen-

value case it is now possible to have several formal solution vectors corresponding to a particular eigenvalue q_{oj} of C_o (see e.g. equation (2.4.22)). Nevertheless, if q_{oj} is an interior point, and not a boundary point of the polygon it will be necessary (if at least one end-point of integration is taken at infinity) to express the error vector by a simultaneous pair of Volterra vector integral equations. If q_{oj} is an extreme point of the polygon and only one formal vector solution corresponds to q_{oj} it is again possible as in the previous section, to express the error vector by a single Volterra integral equation. If q_{oj} is an extreme point of the polygon and more than one formal vector solution corresponds to q_{oj} , or else if q_{oj} is on an edge of the polygon (and not necessarily an extreme point) we can always determine whether we can express the error vector by a single or a simultaneous pair of Volterra integral equations by testing whether or not the above generalized extreme eigenvalue conditions can be satisfied.

3.4.3 General Treatment of $R_{ijm}(z)$

We shall accurately estimate the integral

$$(3.4.10) \quad R_{ijm}^*(z) = - \int_{\zeta}^z e^{Q_i(z)-Q_i(t)} R_{ijm}(t) e^{Q_j(t)} dt$$

where $R_{ijm}(t)$ is given by equation (3.4.5) or (3.4.6), and the contour integral is taken along any one of the paths specified in Section 3.4.2.

In order to conveniently evaluate the dominating term in this integral we write (3.4.5) in the form

$$(3.4.11) \quad R_{ijm}(t) = [Q_i'(t)U_{ijm} + \frac{m}{t} U_{ijm} - U_{ijm} Q_j'(t)] t^{-m} \\ + P_{ijm}(t) - \sum_{k=0}^{m-1} \{B_{m-k}(t)U_{jk}\}_i t^{-m-2}$$

where

$$(3.4.12) \quad P_{ijm}(t) = - \left([(q_{i,p+1} - q_{j,p+1})t^{r-p-1} + \dots + (q_{i,r+1} - q_{j,r+1})t^{-1} + mt^{-1} \right. \\ \left. + J_{\mu_i} t^{-1}] U_{ijm} - U_{ijm} J_{\mu_j} t^{-1} \right) t^{-m} \\ - \sum_{\mu=m+p+1}^{m+r+1} \left[\sum_{k=\mu+1-m}^{\min(r+1,\mu)} (q_{ik} - q_{jk}) U_{ij,\mu-k} J_{\mu_i} U_{ij,\mu-r-1} - U_{ij,\mu-r-1} J_{\mu_j} \right. \\ \left. + (\mu-r-1)U_{ij,\mu-r-1} + \sum_{k=0}^{\mu-r-2} \{B_k U_{j,\mu-r-2-k}\}_i \right] t^{r-\mu}$$

The first term in (3.4.11) is already in a form such that it can be easily bounded. Assuming that for some integer p in $0 \leq p \leq r$ we have $q_{ip} \neq q_{jp}$, let

$$(3.4.13) \quad P_{ijm}^*(z) = \int_{\xi}^z e^{Q_i(z)-Q_i(t)} P_{ijm}(t) e^{Q_j(t)} dt$$

where $P_{ijm}(t)$ is given in (3.4.12). It is convenient to define $S_{ij}(z,t)$ by

$$(3.4.14) \quad S_{ij}(z,t) e^{q_{ij}(z)-q_{ij}(t)} = e^{Q_i(z)-Q_i(t)} P_{ijm}(t) e^{Q_j(t)-q_j(z)}.$$

Clearly, knowing the form of $Q_i(t)$ the matrix $S_{ij}(z,t)$ can be explicitly expressed. In any case $S_{ij}(z,t)$ is nothing more than $P_{ijm}(t)$ multiplied by a matrix polynomial in $\log \left(\frac{z}{t} \right)$. Thus

$$(3.4.15) \quad P_{ij}^*(z) e^{-q_j(z)} = \int_{\zeta}^z \frac{1}{-q'_{ij}(t)} S_{ij}(z,t) e^{q_{ij}(z)-q_{ij}(t)} (-q'_{ij}(t) dt) \\ = - \frac{e^{q_{ij}(z)-q_{ij}(t)}}{q'_{ij}(t)} S_{ij}(z,t) \Big|_{\zeta}^z + \int_{\zeta}^z \frac{\partial}{\partial t} \left(\frac{S_{ij}(z,t)}{q'_{ij}(t)} \right) e^{q_{ij}(z)-q_{ij}(t)} dt ,$$

so that, if for some integer p in $0 \leq p \leq r$ we have $q_{ip} - q_{jp} \neq 0$, then

$$(3.4.16) \quad R_{ijm}^*(z) = e^{Q_i(z)-Q_i(t)} U_{ijm} t^{-m} e^{Q_j(t)} \Big|_{t=\zeta}^{t=z} \\ + \frac{e^{q_i(z)-q_{ij}(t)}}{q'_{ij}(t)} S_{ij}(z,t) \Big|_{t=\zeta}^{t=z} \\ - \int_{\zeta}^z \left[\frac{\partial}{\partial t} \left(\frac{S_{ij}(z,t)}{q'_{ij}(t)} \right) e^{q_i(z)-q_{ij}(t)} - e^{Q_i(z)-Q_i(t)} \sum_{k=0}^{m-1} \{B_{m-k}(t) U_{jk}\}_i t^{-m-2} \right. \\ \left. e^{Q_i(t)} \right] dt$$

On the other hand it follows readily from (3.4.6), that if $i = j$, or if $i \neq j$ but $q_{ip} = q_{jp}$ ($0 \leq p \leq r$) then

$$(3.4.17) \quad R_{ijm}^*(z) = e^{Q_i(z)-Q_i(t)} U_{ijm} t^{-m} e^{Q_j(t)} \Big|_{t=\zeta}^{t=z} \\ + \int_{\zeta}^z e^{Q_i(z)-Q_i(t)} \sum_{k=0}^{m-1} \{B_{m-k}(t) U_{jk}\}_i t^{-m-2} e^{Q_j(t)} dt$$

The right hand sides of both equations (3.4.16) and (3.4.17) are now in a form easy to bound, under the conditions of the paths specified in the previous section.

In the next two sections we shall obtain the error bounds. The analysis is carried out under the assumption that end points are fixed bounded or unbounded points.

Considerations which motivate our procedure of bounding the elements $R_{ijm}^*(z) e^{-q_j(z)}$ (equations (3.4.16) and (3.4.17)) are the following. We observe that each element of $R_{ijm}^*(z) e^{-q_j(z)}$ is of the form

$$(3.4.18) \quad \rho(z, \zeta) = f(z) - f(\zeta) + g(z) - g(\zeta) + \int_{\zeta}^z h(z, t) t^{-m-2} dt$$

where $f(z) \leq O(|z|^{-m} |\log z|^{\mu-\nu})$ (see equation (2.4.22)), $g(z) \leq O(|z|^{-m-1} |\log z|^{\mu-\nu})$ and $|h(z, t)| \leq O(|\log z|^k |\log t|^{\ell}) |z|$, $|t| \rightarrow \infty$, where k and ℓ are non-negative integers. The reason for introducing terms like $t^{-\eta}$, $t^{-\eta_1}$ where η and η_1 are non-negative numbers less than one is to conveniently eliminate logarithmic terms while at the same time obtain good bounds. Hence

$$(3.4.19) \quad |\rho(z, \zeta)| \leq \left| \frac{f(z) - f(\zeta)}{z^{\eta_1-m} - \zeta^{\eta_1-m}} \right| |z^{\eta_1-m} - \zeta^{\eta_1-m}| \\ + \left| \frac{g(z) - g(\zeta)}{z^{\eta_1-m-1} - \zeta^{\eta_1-m-1}} (z^{\eta_1-m-1} - \zeta^{\eta_1-m-1}) + \int_{\zeta}^z h(z, t) t^{-\eta_1} t^{\eta_1-m-2} dt \right|$$

$$\begin{aligned}
 |\rho(z, \zeta)| &\leq \left| \frac{f(z) - f(\zeta)}{z^{\eta_1-m} - \zeta^{\eta_1-m}} \right| \left| (\eta_1 - m) \int_{\zeta}^z t^{\eta_1-m-1} dt \right| \\
 &+ \left| \int_{\zeta}^z \left\{ \frac{(\eta_1-m-1)(g(z)-g(\zeta))}{z^{\eta_1-m-1} - \zeta^{\eta_1-m-1}} + h(z, t)t^{-\eta_1} \right\} t^{\eta_1-m-2} dt \right| \\
 &\leq \left\{ \sup_{\tau \in \mathcal{D}} \left| \frac{f(\tau) - f(\zeta)}{\tau^{\eta_1-m} - \zeta^{\eta_1-m}} \right| \right\} \mathcal{V}_{\mathcal{D}}(t^{\eta_1-m}) \\
 &+ \left\{ \sup_{t, \tau \in \mathcal{D}} \left| \frac{g(\tau) - g(\zeta)}{\tau^{\eta_1-m-1} - \zeta^{\eta_1-m-1}} + \frac{h(\tau, t)t^{-\eta_1}}{\eta_1 - m - 1} \right| \right\} \mathcal{V}_{\mathcal{D}}(t^{\eta_1-m-1})
 \end{aligned}$$

where in the last limit superior t and τ are points on $\mathcal{D}$ in the order ζ, t, τ, z .

Thus let η_1 be a fixed positive number which we take to be zero if $J_{\mu_i} = 0$ ($i = 1, 2, \dots, \ell$) and in $0 < \eta_1 < 1$ if at least one $J_{\mu_i} \neq 0$. It is convenient to define vectors $U_{jm}^{v*}(\tau)$, $V_{jm}^{v1}(\tau)$ and $V_{jm}^{v2}(\tau, t)$ by

$$\begin{aligned}
 U_{ijm}^{\omega v*}(\tau) &= \frac{\left\{ U_{ijm} \tau^{-m} e^{Q_j(\tau)-q_j(\tau)} - e^{Q_i(\tau)-Q_i(\zeta)} U_{ijm} \zeta^{-m} e^{Q_j(\zeta)-q_j(\tau)} \right\}^{\omega v}}{\tau^{\eta_1-m} - \zeta^{\eta_1-m}} \quad (i = 1, 2, \dots, \ell) \\
 V_{ijm}^{\omega v1}(\tau) &= \frac{\left\{ S_{ij}(\tau, \tau)/q'_{ij}(\tau) - e^{q_{ij}(\tau)-q_{ij}(\zeta)} S_{ij}(\tau, \zeta)/q'_{ij}(\zeta) \right\}^{\omega v}}{\tau^{\eta_1-m-1} - \zeta^{\eta_1-m-1}} \quad (i \neq j)^* \\
 (3.4.20) \quad &= 0 \quad \text{otherwise;}
 \end{aligned}$$

$$\begin{aligned}
 V_{ijm}^{\omega v2}(\tau, t) &= \frac{t^{-\eta_1}}{m+1-\eta_1} \left\{ \frac{\partial}{\partial t} \left(\frac{S_{ij}(\tau, t)}{q'_{ij}(t)} \right) e^{q_{ij}(\tau)-q_{ij}(t)} t^{m+2} + \right. \\
 &\quad \left. - e^{Q_i(\tau)-Q_i(t)} \sum_{k=1}^{m-1} \{ B_{m-k}(t) U_{jk} \}_k e^{Q_j(t)-q_j(\tau)} \right\}^{\omega v} \quad (i \neq j)^* \\
 &\quad \frac{t^{-\eta_1}}{m+1-\eta_1} \left\{ e^{Q_j(\tau)-Q_j(t)} \sum_{k=0}^{m-1} \{ B_{m-k}(t) U_{jk} \}_i e^{Q_j(t)-q_j(\tau)} \right\}^{\omega v} \\
 &\quad \text{otherwise.}
 \end{aligned}$$

In (3.4.20) $\{A_j\}_i^{\omega\nu} = A_{ij}^{\omega\nu}$ indicates the (ω, ν) 'th element in the (i, j) 'th block of the matrix A , and the notation $(i \neq j)^*$ to the right of a particular equation indicates that the particular equation contains all those i 'th elements for which there is an integer p in $0 \leq p \leq r$ such that $q_{ip} - q_{jp} \neq 0$.

3.4.4 Error Bounds for Extreme Eigenvalue and Related Cases

If a block $\eta_{jm}(z)$ of μ_j vectors of holomorphic functions satisfies

$$(3.4.21) \quad \eta_{jm}(z) = \int_{\xi}^z \exp[Q(z) - Q(t)] \{t^{-2} B(t) \eta_{jm}(t) - R_{jm}(t) e^{Q_j(t)}\} dt$$

where the path $\mathcal{P}$ of integration satisfies the extreme eigenvalue conditions in Section 3.4.2, then it follows that $\eta_{jm}(z)$ also satisfies (3.4.8).

Let us put $\eta_{jm}(z) = \epsilon_{jm}^v(z) e^{q_j(z)}$, and let us consider the v 'th column vector $\epsilon_{jm}^v(z)$ of $\epsilon_{jm}(z)$. With reference to the discussion surrounding the equation (2.4.22), the integral equation for $\epsilon_{jm}^v(z)$ takes the form

$$(3.4.22) \quad \epsilon_{jm}^v(z) = \int_{\xi}^z e^{D_j(z) - D_j(t)} \left\{ t^{-2} B(t) \epsilon_{jm}^v(t) - \sum_{s=v}^{\mu} R_{jm}^s(t) \frac{(\log t)^{s-v}}{(s-v)!} \right\} dt$$

where $R_{jm}^s(t)$ is the s 'th column vector ($s = 1, 2, \dots, \mu_j$) in $R_{jm}(t)$, and $D_j(z) = Q(z) - I q_j(z)$.

Thus, using (3.4.20) and taking t and τ in the order ξ, t, τ, z on $\mathcal{P}$ we define

$$B = \frac{1}{1-\eta} \left[\sup_{t, \tau \in \mathcal{D}} \left| \left\{ e^{D_j(\tau) - D_j(t)} B(t) t^{-\eta} \right\}_{ks} \right| \right]$$

$$(3.4.23) \quad U_{ijm}^{\omega\nu*} = \sup_{\tau \in \mathcal{D}} |U_{ijm}^{\omega\nu*}(\tau)|$$

$$\gamma_{ij}^{\omega\nu} = \sup_{t, \tau \in \mathcal{D}} \left\{ |v_{ijm}^{\omega\nu 1}(\tau) + v_{ijm}^{\omega\nu 2}(\tau, t)| \right\}$$

where the notation $\{A\}_{ks}$ indicates the (k,s) 'th element of the matrix A , and the fixed number η is zero if J_{μ_j} is zero but is otherwise an arbitrary positive number less than 1.

We are now in position to invoke the analysis of Section B.1 of Appendix B to obtain

Theorem 3.4.1 If corresponding to a formal vector solution $\tilde{W}_j^\nu(z)$ of equation (2.4.4) we can define a region $\mathcal{D}(z, \zeta)$ as described by the extreme eigenvalue conditions in Section 3.4.2, then the equation (2.4.4) possesses an actual solution vector

$$(3.4.24) \quad W_{jm}^\nu(z) = \left[\sum_{k=0}^{m-1} \left(\sum_{s=\nu}^{\mu} U_{jk}^s \frac{(\log z)^{s-\nu}}{(s-\nu)!} z^{-k} + \epsilon_{jm}^\nu(z) \right) e^{q_j(z)} \right]$$

of functions holomorphic in $\mathcal{D}(z, \zeta)$ such that

$$(3.4.25) \quad |\epsilon_{jm}^\nu(z)| \leq \exp \left\{ B \gamma_{\mathcal{D}}(t^{\eta-1}) \right\} \left\{ U_{jm}^{\nu*} \gamma_{\mathcal{D}}(t^{\eta_1-m}) + \gamma_{jm}^\nu \gamma_{\mathcal{D}}(t^{\eta_1-m-1}) \right\}$$

for all z in $\mathcal{D}(z, \zeta)$, where the matrix B and the elements of the vectors $U_{jm}^{\nu*}$ and γ_{jm}^ν are defined by equation (3.4.23). The solution $W_{jm}^\nu(z)$ depends on ζ and an arbitrary positive integer m . The function

$q_j(z)$ is defined in Theorem 2.4.1. The integers μ and ν and the vectors U_{jk}^s ($k = 0, 1, 2, \dots; s = 1, 2, \dots, \mu_j$) are defined as for equation (2.4.22). If each $J_{\mu_i} = 0$ ($i = 1, 2, \dots, \ell$) then the numbers η and η_1 above can be taken to be zero; if $J_{\mu_j} = 0$ then η can be taken to be zero; otherwise η and η_1 are arbitrary positive numbers less than 1.

Let us now obtain a norm bound for the vector $\epsilon_{jm}^\nu(z)$ in Theorem 3.4.1. To achieve this goal we assume the definitions (3.4.20) for the vectors $U_{jm}^{\nu*}(z)$, $V_{jm}^{\nu 1}(z)$, $V_{jm}^{\nu 2}(z, t)$, and define

$$B = \frac{1}{1 - \eta} \sup_{t, \tau \in \mathcal{P}} \|e^{D_j(\tau) - D_j(t)} B(t) t^{-\eta}\|$$

$$(3.4.26) \quad U_{jm}^{\nu*} = \sup_{\tau \in \mathcal{P}} \|U_{jm}^{\nu*}(\tau)\|$$

$$\gamma_{jm}^\nu = \sup_{t, \tau \in \mathcal{P}} \|V_{jm}^{\nu 1}(\tau) + V_{jm}^{\nu 2}(\tau, t)\|$$

where t and τ are on $\mathcal{P}$ in the order ζ, t, τ, z .

Theorem 3.4.2 A norm bound for the vector $\epsilon_{jm}^\nu(z)$ in Theorem 3.4.1 is given by

$$(3.4.27) \quad \|\epsilon_{jm}^\nu(z)\| \leq \exp \left\{ B \mathcal{V}_{\mathcal{P}}(t^{\eta-1}) \right\} \left\{ U_{jm}^{\nu*} \mathcal{V}_{\mathcal{P}}(t^{\eta_1-m}) + \gamma_{jm}^\nu \mathcal{V}_{\mathcal{P}}(t^{\eta_1-m-1}) \right\}$$

where the numbers B , $U_{jm}^{\nu*}$ and γ_{jm}^ν are defined by equation (3.4.26).

The numbers η and η_1 are defined as in Theorem 3.4.1.

3.4.5 Error Bounds for Interior Eigenvalue and Related Cases

Substituting $\eta_{jmk}(z) = \epsilon_{jmk}^\nu(z) e^{q_j(z)}$ ($k = 1, 2$) in (3.4.9) and integrating as in (3.3.13) and (3.3.28) we obtain

$$(3.4.28) \quad \begin{bmatrix} \epsilon_{jm1}(z) \\ \epsilon_{jm2}(z) \end{bmatrix} = \begin{bmatrix} \int_{\zeta_1}^z & 0 \\ 0 & \int_{\zeta_2}^z \end{bmatrix} \begin{bmatrix} e^{D_{j1}(z)-D_{j1}(t)} & 0 \\ 0 & e^{D_{j2}(z)-D_{j2}(t)} \end{bmatrix} t^{-2} .$$

$$\cdot \begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix} \begin{bmatrix} \epsilon_{jm1}(t) \\ \epsilon_{jm2}(t) \end{bmatrix} dt + \begin{bmatrix} R_{jm1}^*(z) \\ R_{jm2}^*(z) \end{bmatrix} e^{-q_j(z)}$$

where in the notation of (3.4.9)

$$(3.4.29) \quad R_{jmk}^*(z) = - \int_{\zeta_k}^z e^{D_{jk}(z)-D_{jk}(t)} R_{jmk}(t) e^{Q_j(t)} dt \quad (k = 1,2).$$

For purposes of bounding the solution of (3.4.28) it is again convenient to use the definitions (3.4.20). We thus partition each of the vectors $U_{jm}^{v*}(\tau)$, $V_{jm}^{v1}(\tau)$, $V_{jm}^{v2}(\tau, t)$ into two lower dimensional vectors $U_{jmk}^{v*}(\tau)$, $V_{jmk}^{v1}(\tau)$, $V_{jmk}^{v2}(\tau, t)$ ($k = 1,2$) in a manner analogous to the way in which $R_{jm}^*(z)$ was split into lower dimensional vectors $R_{jmk}^*(z)$ ($k = 1,2$). It is of course presumed that the point ζ in (3.4.20) is appropriately changed to ζ_k if $i \in N_k$ ($k = 1,2$; see the interior eigenvalue conditions of Section 3.4.2).

We shall first obtain a norm bound for the solution of (3.4.28).

It is thus convenient to define scalar functions $\psi_{mk}(\cdot, \cdot)$ by

$$\begin{aligned}
 \psi_{m1}(\zeta_1, z) = & \sup_{t_2 \in \mathcal{P}} \|U_{jm1}^{\nu*}(t_2)\| \mathcal{V}_{\zeta_1, z}(t^{\eta_1-m}) + \\
 (3.4.30) \quad & + \sup_{t_1, t_2 \in \mathcal{P}} \left\{ \|V_{jm1}^{\nu1}(t_2) + V_{jm2}^{\nu2}(t_2, t_1)\| \right\} \mathcal{V}_{\zeta_1, z}(t^{\eta_1-m-1})
 \end{aligned}$$

$$\begin{aligned}
 \psi_{m2}(z, \zeta_2) = & \sup_{t_1 \in \mathcal{P}} \|U_{jm2}^{\nu*}(t_1)\| \mathcal{V}_{z, \zeta_2}(t^{\eta_1-m}) + \\
 & + \sup_{t_1, t_2 \in \mathcal{P}} \left\{ \|V_{jm2}^{\nu1}(t_1) + V_{jm2}^{\nu2}(t_1, t_2)\| \right\} \mathcal{V}_{z, \zeta_2}(t^{\eta_1-m-1})
 \end{aligned}$$

where t_1 and t_2 are located on $\mathcal{P}$ in the order $\zeta_1, t_1, t_2, \zeta_2$ and the fixed number η_1 is chosen[†] as for (3.4.20).

Similarly, with t_1 and t_2 chosen as for (3.4.30) and η zero if J_{μ_j} is zero but otherwise η an arbitrary positive number less than 1, we define non-negative numbers α, β, γ and δ by

$$\begin{aligned}
 \alpha = & \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{t_1^{-\eta}}{1-\eta} e^{D_{j1}(t_2)-D_{j1}(t_1)} \alpha(t_1) \right\|, \\
 \beta = & \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{t_1^{-\eta}}{1-\eta} e^{D_{j1}(t_2)-D_{j1}(t_1)} \beta(t_1) \right\|, \\
 (3.4.31) \quad \gamma = & \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{t_2^{-\eta}}{1-\eta} e^{D_{j2}(t_1)-D_{j2}(t_2)} \gamma(t_2) \right\|, \\
 \delta = & \sup_{t_1, t_2 \in \mathcal{P}} \left\| \frac{t_2^{-\eta}}{1-\eta} e^{D_{j2}(t_1)-D_{j2}(t_2)} \delta(t_2) \right\|.
 \end{aligned}$$

[†] A variation is of course possible, i.e. we could choose one particular η for ψ_{m1} and a different η for ψ_{m2} . For simplicity of expression we have used one η in (3.4.30) as well as in (3.4.31).

On combining the above definitions with Theorem B.2 of Appendix B we obtain

Theorem 3.4.3 If corresponding to a formal solution vector $\tilde{W}_{jm}^v(z)$ (c.f. equation 2.4.22)) we can define a region $\mathcal{D}(\zeta_1, \zeta_2, \eta)$ as described by the interior eigenvalue conditions in Section 3.4.2, then the equation (2.4.4) possesses an actual solution vector

$$(3.4.32) \quad \begin{pmatrix} W_{jm1}^v(z) \\ W_{jm2}^v(z) \end{pmatrix} = \left[\sum_{k=0}^{m-1} \left[\sum_{s=v}^{\mu} \begin{pmatrix} U_{jk1}^s \\ U_{jk2}^s \end{pmatrix} \frac{(\log z)^{s-v}}{(s-v)!} \right] z^{-k} + \begin{pmatrix} \epsilon_{jm1}^v(z) \\ \epsilon_{jm2}^v(z) \end{pmatrix} e^{q_j(z)} \right]$$

of functions holomorphic in $\mathcal{D}(\zeta_1, \zeta_2, \eta)$ such that

$$(3.4.33) \quad \begin{Bmatrix} \|\epsilon_{jm1}^v(z)\| \\ \|\epsilon_{jm2}^v(z)\| \end{Bmatrix} \leq \begin{Bmatrix} \psi_{m1}(\zeta_1, z) \\ \psi_{m2}(z, \zeta_2) \end{Bmatrix} + \begin{Bmatrix} \gamma_{\zeta_1, z}(t^{\eta-1}) & 0 \\ 0 & \gamma_{z, \zeta_2}(t^{\eta-1}) \end{Bmatrix} E f[E \gamma_p(t^{\eta-1})] \begin{Bmatrix} \psi_{m1}(\zeta_1, \zeta_2) \\ \psi_{m2}(\zeta_1, \zeta_2) \end{Bmatrix}$$

where $E f(E\chi)$ is defined by equation (3.3.33). The elements in the solution vector $W_{jmk}^v(z)$ depend on ζ_k ($k = 1, 2$) and an arbitrary positive integer m . The function $q_j(z)$ is defined as in Theorem 2.4.1. The vector U_{jk}^s is the s^{th} column vector in the j^{th} column of blocks of U_k (defined in the proof of Theorem 2.4.2); it is partitioned into lower dimensional vectors as described in Section 3.4.2. The functions $\psi_{jmk}(\cdot, \cdot)$ ($k = 1, 2$) are defined by (3.4.30); η_1 is an arbitrary positive number less than one, or zero, depending upon whether or not

there is at least one $J_{\mu_i} \neq 0$ ($i = 1, 2, \dots, \ell$). The non-negative numbers α, β, γ and δ are defined by equation (3.4.31).

In order to obtain a vector bound for the vectors $\epsilon_{jm1}^v(z)$ and $\epsilon_{jm2}^v(z)$ in equation (3.4.28) we put $\alpha(t) = [\{\alpha(t)\}_{sh} \ (s, h = 1, 2, \dots, \kappa)]$;
 $\beta(t) = [\{\beta(t)\}_{sh} \ (s = 1, 2, \dots, \kappa; \ h = \kappa + 1, \kappa + 2, \dots, n)]$;
 $\gamma(t) = [\{\gamma(t)\}_{sh} \ (s = \kappa + 1, \kappa + 2, \dots, n; \ h = 1, 2, \dots, \kappa)]$;
 $\delta(t) = [\{\delta(t)\}_{sh} \ (s, h = \kappa + 1, \kappa + 2, \dots, n)]$ where by our partitioning of the matrix $B(t)$ into blocks $\beta(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$, the matrix $\alpha(t)$ is $\kappa \times \kappa$, and similarly for $\beta(t)$, $\gamma(t)$ and $\delta(t)$ as indicated here. With η defined as in the above theorem we define matrices α, β, γ and δ of non-negative elements by

$$\begin{aligned} \alpha &= \frac{1}{1 - \eta} \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ t_1^{-\eta} e^{D_{j1}(t_2) - D_{j1}(t_1)} \alpha(t_1) \}_{sh} | \right] \\ \gamma &= \frac{1}{1 - \eta} \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ t_1^{-\eta} e^{D_{j1}(t_2) - D_{j1}(t_1)} \beta(t_1) \}_{sh} | \right] \\ \beta &= \frac{1}{1 - \eta} \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ t_2^{-\eta} e^{D_{j2}(t_1) - D_{j2}(t_2)} \gamma(t_2) \}_{sh} | \right] \\ \delta &= \frac{1}{1 - \eta} \left[\sup_{t_1, t_2 \in \mathcal{P}} | \{ t_2^{-\eta} e^{D_{j2}(t_1) - D_{j2}(t_2)} \delta(t_2) \}_{sh} | \right] \end{aligned} \quad (3.4.34)$$

where t_1 and t_2 are defined as for equation (3.4.31), $[\{\alpha(t_1)\}_{sh}] = \alpha(t_1) = [\{\alpha(t_1)\}_{sh} \ (s, h = 1, 2, \dots, \kappa)]$ and similarly for $\beta(t_1)$, $\gamma(t_2)$ and $\delta(t_2)$ as above.

Next, with the vectors $U_{jmk}^{v*}(\tau)$, $V_{jmk}^{v1}(\tau)$ and $V_{jmk}^{v2}(\tau, t)$ ($k=1, 2$) defined as for equation (3.4.30) we define vectors $\psi_{mk}(\ , \)$ ($k = 1, 2$) of non-negative elements by

$$(3.4.35) \quad \begin{aligned} \psi_{m1}(\zeta_1, z) = & \sup_{t_2 \in \mathcal{P}} |U_{jm1}^{*v}(t_2)| \gamma_{\mathcal{P}\zeta_1, z}(t^{\eta_1-m}) \\ & + \sup_{t_1, t_2 \in \mathcal{P}} \left\{ |v_{jm1}^{v1}(t_2) + v_{jm1}^{v2}(t_2, t_1)| \right\} \gamma_{\zeta_1, z}(t^{\eta_1-m-1}) \end{aligned}$$

$$\begin{aligned} \psi_{m2}(z, \zeta_2) = & \sup_{t_2 \in \mathcal{P}} |U_{jm2}^{*v}(t_2)| \gamma_{z, \zeta_2}(t^{\eta_1-m}) \\ & + \sup_{t_1, t_2 \in \mathcal{P}} \left\{ |v_{jm2}^{v1}(t_1) + v_{jm2}^{v2}(t_1, t_2)| \right\} \gamma_{z, \zeta_2}(t^{\eta_1-m-1}), \end{aligned}$$

where the limit superiors are taken over each individual element of the vectors. The number η_1 and the points t_1 and t_2 are as described for equation (3.4.30).

The remaining details concerning the proof of the following theorem are carried out in Appendix C.

Theorem 3.4.4 If all eigenvalues of the matrix $C_2 C_1(\zeta_1, \zeta_2)$ (see equation (3.4.37)) are less than one in magnitude then the vector $\epsilon_{jm1}^v(z)$ (equation (3.4.32) is bounded as

$$(3.4.36) \quad \begin{aligned} |\epsilon_{jm1}(z)| \leq & \exp \left\{ \alpha \gamma_{\zeta_1, z}(t^{\eta-1}) \right\} \psi_{m1}(\zeta_1, z) \\ & + C_1(\zeta_1, z) \left[\exp \left\{ \delta \gamma_{\mathcal{P}}(t^{\eta-1}) \right\} \psi_{m2}(\zeta_1, \zeta_2) + B \right] \end{aligned}$$

where

$$B = [I_2 - C_2 C_1(\zeta_1, \zeta_2)]^{-1} C_2.$$

$$(3.4.37) \quad [\exp \{\delta \gamma_{\rho}(t^{\eta-1})\} \psi_{m1}(\zeta_1, \zeta_2) + C_1(\zeta_1, \zeta_2) \exp \{\delta \gamma_{\rho}(t^{\eta-1})\} \psi_{m2}(\zeta_1, \zeta_2)]$$

$$C_1(\zeta_1, z) = \alpha^{-1} [\exp \{\alpha \gamma_{\zeta_1, z}(t^{\eta-1})\} - I_1] \beta; \quad C_2 = \delta^{-1} [\exp \{\delta \gamma_{\rho}(t^{\eta-1})\} - I_2]$$

and where I_1 designates a $\kappa \times \kappa$ unit matrix, and I_2 designates an $(n - \kappa) \times (n - \kappa)$ unit matrix, κ^2 being the number of elements in $\alpha(t)$ (equation (3.4.9)). The vectors $\psi_{mk}(\cdot, \cdot)$ ($k = 1, 2$) are defined by equation (3.4.35), while the matrices α, β, γ and δ are defined by (3.4.34). The numbers η and η_1 are defined as in Theorem 3.4.3.

A norm bound which is sharper in the neighborhood of infinity than that given in Theorem 3.4.3 can be obtained by a procedure exactly similar to that for the above theorem. The proof is given in Section C.3 of Appendix C.

Theorem 3.4.5 With the notation of Theorem 3.4.3 let $\Omega = \max(\alpha, \delta)$.

If

$$(3.4.38) \quad C = \frac{\beta \gamma}{\Omega(\alpha + \beta)} \exp \left\{ \left(\Omega + \frac{|\alpha - \delta|}{2} \right) \gamma_{\rho}(t^{\eta-1}) \right\} \sinh \left\{ \frac{\alpha + \delta}{2} \gamma_{\rho}(t^{\eta-1}) \right\} < 1$$

then the norm of the vector $\epsilon_{jm1}^v(z)$ (equation (3.4.32)) is bounded as

$$(3.4.39) \quad \|\epsilon_{jm1}^v(z)\| \leq \exp \left\{ \alpha \gamma_{\zeta_1, z}(t^{\eta-1}) \right\} \psi_{m1}(\zeta_1, z) \\ + Y(z) \left\{ \psi_{m2}(\zeta_1, \zeta_2) + \gamma(1 - C)^{-1} \left[D(a) \psi_{m1}(\zeta_1, \zeta_2) + E \psi_{m2}(\zeta_1, \zeta_2) \right] \right\}$$

where

$$Y(z) = \frac{2\beta}{\alpha+\delta} \exp \left\{ \left(\Omega + \frac{|\alpha-\delta|}{2} \right) \mathcal{V}_{\rho}(t^{\eta-1}) \right\} \sinh \left\{ \frac{\alpha+\delta}{2} \mathcal{V}_{\zeta_1, z}(t^{\eta-1}) \right\}$$

$$(3.4.40) \quad D(\chi) = \frac{1}{\chi} \left[\exp \left\{ \chi \mathcal{V}_{\rho}(t^{\eta-1}) \right\} - 1 \right]$$

$$E = \frac{\beta}{\alpha+\delta} \left[D(\alpha + \delta) - 2D(0) + D(\alpha) \exp \left\{ \delta \mathcal{V}_{\rho}(t^{\eta-1}) \right\} \right]$$

CHAPTER IV

CONCLUSION

In this chapter we summarize the contents of the thesis, and state some unsolved problems.

In this thesis we have found error bounds for partial sum approximations of formal solutions to actual solutions of the system of differential equations

$$(4.1) \quad \frac{dX}{dt} = t^h A(t) X = t^h \left[\sum_{k=0}^{s-1} A_k t^{-k} + A_s(t) t^{-s} \right] X \quad (s = 1, 2, \dots)$$

where the n^2 elements of the $n \times n$ matrix $A_s(t)$ ($s = 0, 1, 2, \dots$) are holomorphic and bounded in a domain $\mathcal{Q}$ (which may be a Riemann surface) extending to infinity and where h is an integer.

The first step in solving (4.1) is to transform this equation to canonical form. In this connection, the existence of an $n \times n$ matrix polynomial

$$(4.2) \quad T(t) = \sum_{k=0}^N T_k t^{-k/p}$$

where p and N are positive integers and the T_k 's ($k = 0, 1, \dots, N$) are $n \times n$ matrices such that when the transformation $X = T(t)W$ is made in (4.1) the resulting system has the desired canonical form (see Section 2.4) is established in [32].

In the majority of the cases when $h \geq 0$ fractional powers will not be required in the transformation to canonical form; instead $T(t)$

will be of the form

$$(4.3) \quad T(t) = \sum_{k=0}^{r+1} T_k t^{-k} \quad (h = r)$$

with T_k ($k = 0, 1, \dots, r+1$) an $n \times n$ constant matrix. The resulting transformed system has distinct eigenvalues in the lead coefficient matrix, and has an irregular singularity of rank $r + 1 = h + 1$ at infinity. Other possibilities can however also arise; well-known ones among these are the case when the coefficient matrix has a regular singular point at infinity ($r = -1$), or the case when the coefficient matrix has a regular point at infinity ($r \leq -2$). In our transformations to canonical form in Chapter II of the thesis we have given special consideration to each of the above cases; we have explicitly constructed transformations which transform the system (4.1) to canonical form in the case when $h = -1$, and in the case when $h \geq 0$ and the lead coefficient matrix (A_0) has distinct eigenvalues. For the very complex general case we have merely stated the results in [32]. We have however proved modifications of the usual canonical forms for the general case and for the case when $h = -1$, which are as easy to achieve as the usual forms. In addition, these modifications are an aid to obtaining error bounds in Chapter III. For present purposes it is convenient to denote the general transformed system by

$$(4.4) \quad \frac{dW}{dz} = z^r C(z)W = z^r \left[\sum_{k=0}^{s-1} C_k z^{-k} + C_s(z) z^{-s} \right] W \quad (s = 1, 2, \dots)$$

where r is an integer and $C(z)$ has the properties described for $A(z)$ (see Section 1.5).

Following transformation to canonical form we have explicitly constructed formal independent series solutions for the transformed system in all cases. These formal series do not in general converge; in the cases when they do converge the formal series solutions are actual solutions of the transformed system. In particular, if in (4.4) $r \leq -1$ and the coefficient matrix $C(z)$ has a power series expansion which converges in some neighborhood of infinity then the formal series solutions obtained in Chapter II converge in this same neighborhood of infinity and are actual solutions of (4.4).

Although the formal series solutions generally diverge they are far from being useless. It is known for example [8] that in all cases the formal solutions of (4.4) are asymptotic expansions of actual solutions of this equation in certain sectors of infinity. Moreover a partial sum of a formal solution is easy to compute. In Chapter III of the thesis we obtain error bounds for partial sum approximations of formal solutions to actual solutions, in all cases. We have again first given special consideration to the cases $r = -2$, $r = -1$ and to arbitrary non-negative r when the eigenvalues of C_0 are distinct. We then concluded Chapter III by finding error bounds for the general system (4.4) in canonical form; the results of this section thus include the previous special cases together with all other remaining cases. The bounds are in all cases given in two forms; as vector bounds and as norm bounds.

For purposes of easing the difficulties in the analysis, the simplest case of a regular point ($r = -2$) at infinity is considered first. The

case of a regular singular point ($r = -1$) is somewhat more complex; it was thus necessary to further restrict the paths of integration in the Volterra integral equations defining the solutions. In the case of an irregular singular point ($r \geq 0$) when the lead coefficient matrix has distinct eigenvalues two cases distinguished themselves: the case when it was possible to define the error vector by a single Volterra vector integral equation; and the case when we required two simultaneous vector integral equations to define the error. The error bounds are considerably sharper in the former case than in the latter. We have therefore given a geometric criterion to distinguish these two cases.

By our method of constructing formal solutions there is a one-to-one correspondence between the formal solution vectors and the eigenvalues of the lead coefficient matrix. Thus if we consider the eigenvalues of C_0 to be points in the complex plane and to be enclosed there by a strictly convex closed polygon, then* corresponding to an eigenvalue which is an extreme point of the polygon it is possible (but not necessary) to express the error vector by a single Volterra integral equation; corresponding to an eigenvalue which is an interior point and not a boundary point it is necessary to use a simultaneous pair of Volterra vector integral equations. Corresponding to an eigenvalue which is on the edge of the polygon but not an extreme point it may or may not be possible to express the error by a single Volterra vector integral equation.

* We have assumed here that end-points of integration in the Volterra integral equations be taken at infinity.

In all of the above special cases it is possible, under the assumption that the coefficient matrix is holomorphic in a sufficiently large domain, to choose end-points of integration in the integral equations defining the solutions at infinity. This is no longer always possible in the general case, even when the matrix $C(z)$ in (4.4) is holomorphic for all $|z|$ sufficiently large. Nevertheless, although it may be necessary to choose one finite end point of integration it is always possible, even under the assumption that we are constructing actual solutions for which the formal solutions are asymptotic expansions of actual solutions to express the error vector by either a single, or at most a simultaneous pair of Volterra integral equations.

The error bounds are in all cases carefully estimated. The problem yet to be solved is to find, among all the possible paths those that minimize the bounds; in this connection a method for finding the paths which minimize the total variations is given in [6,35].

For sake of illustration we have completely solved the $n = 2$ case in Appendix A of the thesis. This is accomplished by first explicitly illustrating with a flow chart a method of transforming the general $n = 2$ case to canonical form. Formal solutions are then constructed for the transformed system, and error bounds are obtained for formal partial sum approximations to actual solutions of the transformed system. Assuming the domain in which the coefficient matrix is holomorphic to be sufficiently large, it is in this case always possible to obtain actual solutions for which the formal solutions are asymptotic expansions by use of a single Volterra vector integral equation.

Appendix B contains the proof of existence and uniqueness of solutions of integral equations in the thesis.

In Appendix C of the thesis we solve some real variable integral inequalities which are useful for obtaining error bounds.

A difficulty can arise in practical applications if solutions are desired corresponding to eigenvalues in the lead coefficient matrix of the general system which are equal, or nearly equal. The main reason for this difficulty is the problem of transforming a matrix to Jordan canonical form; this problem cannot be satisfactorily solved on a computer in the case when two or more of the eigenvalues of the lead coefficient matrix are equal, or nearly equal, due to the fact that only a finite number of decimal places can be carried.

Now it is shown in [5] (Vol. 1, Mémoire Troisième) that in the case when the coefficient matrix of the system (4.1) has a convergent power series expansion about $t = \infty$, the actual solution matrix of this system which reduces to the unit matrix at an arbitrary point t_0 in the domain in which the coefficient matrix is holomorphic is an entire function of t^{-1} as well as of the matrices $A_0, A_1, A_2, \dots$. Thus a small change in the eigenvalues of A_0 produces only a correspondingly small change in the actual solution matrix. What is the corresponding behaviour of a formal solution matrix when two or more eigenvalues of A_0 approach each other? Perhaps by choosing appropriate functions of the eigenvalues as coefficients, it is possible, as in the case of certain second order differential equations^{*} to take linear

* This has been illustrated to me by Professor E.L. Whitney.

combinations of the formal vector solutions which correspond to the eigenvalues such that a continuous change in the eigenvalues produces a continuous change in the formal vector solutions. A quantitative investigation of this problem would be desirable; the solution of which might very well throw light upon the connecting problem ([47] pages 7, 38) as well as solve the difficulty arising as a result of the fact that it is not always possible in practice to transform a given matrix to its Jordan canonical form.

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APPENDIX A

THE $n = 2$ CASE

In this appendix we shall completely solve the general $n = 2$ case. At the outset we explicitly illustrate by flow chart a method of transforming the general singular $n = 2$ system (equation (A.1)) to canonical form. By applying the results of Chapters II and III of the thesis we then obtain formal solutions for the transformed system and error bounds for partial sum approximations of formal solutions to actual solutions.

Let us assume that we are given the particular system $X' = z^h A(z)X$ (h an integer) or

$$(A.1) \quad \begin{bmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{bmatrix} = z^h \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

where for each arbitrary positive integer s

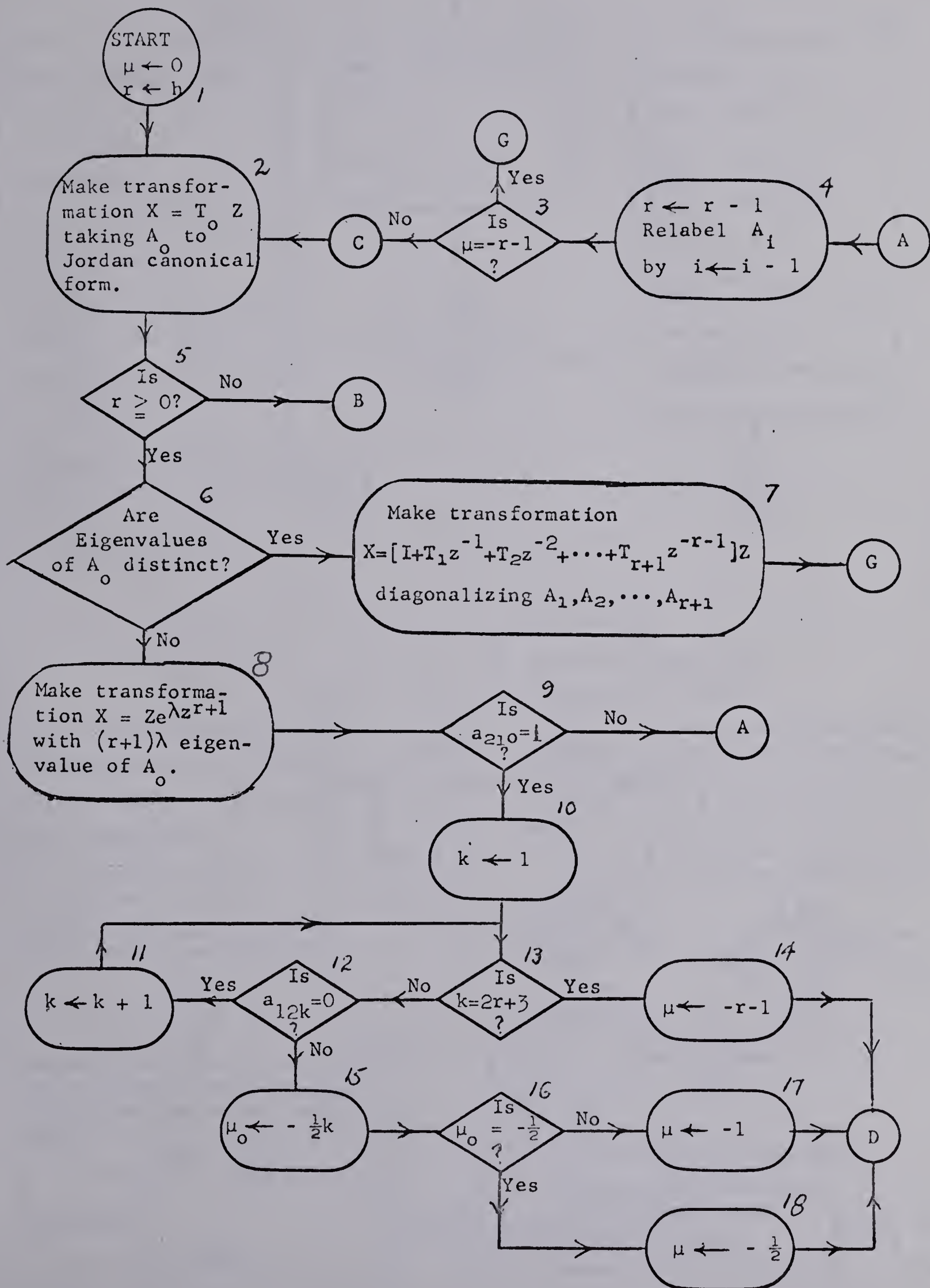
$$(A.2) \quad a_{ij}(z) = \sum_{k=0}^{s-1} a_{ijk} z^{-k} + a_{ijs}(z) z^{-s} \sim \sum_{k=0}^{\infty} a_{ijk} z^{-k} \quad (i, j = 1, 2)$$

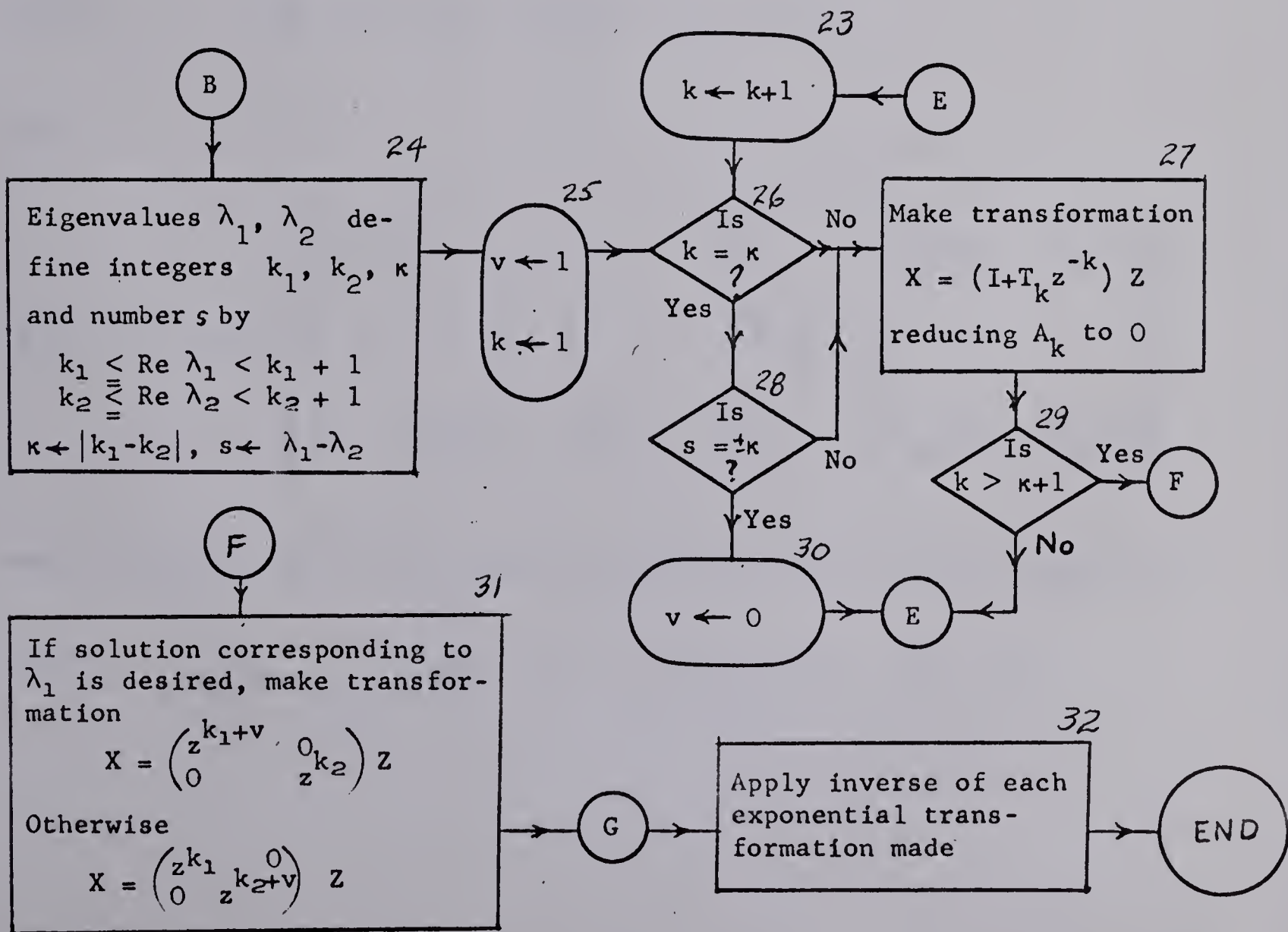
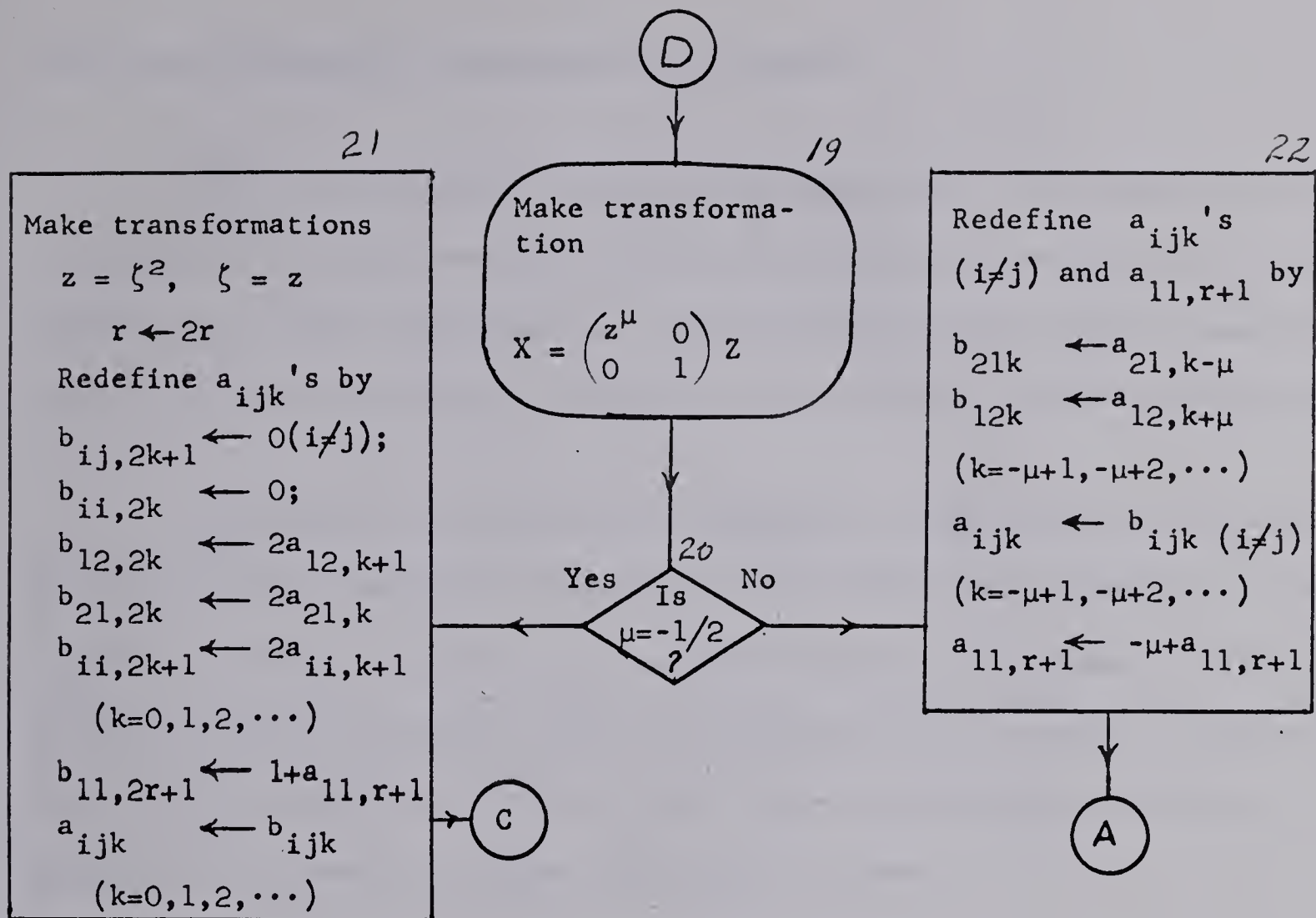
and where we assume that the functions $a_{ij}(z)$ ($i, j = 1, 2$) are holomorphic and bounded in some domain $\mathcal{D}$ extending to infinity.

We begin by transforming the system (A.1) to canonical form. The flow chart on the following pages indicates the steps involved in transforming the most general singular $n = 2$ case to canonical form.

The chart is constructed along the lines of the proof of Turrittin's theorem [32], and therefore contains all the transformations which Turrittin used to prove his theorem for arbitrary n . It follows by inspection that

FLOW CHART ILLUSTRATING TRANSFORMATION OF $n = 2$ SYSTEM
TO CANONICAL FORM





the process eventually terminates in all cases.

Box #2 in the chart contains the normalizing transformation, box #8 the exponential transformation, box #19 the shearing transformation, box #31 the root equalizing transformation, and box #27 the zero inducing transformation. Box #7 also contains a modified form of the zero inducing transformation.

Although it is evident by studying the above flow chart that the desired canonical form can be achieved without use of the exponential transformation, we have included this transformation for two reasons: firstly, it makes the above procedure easier to understand, and secondly, we wanted the chart to resemble the proof of Turrittin's [32] theorem as much as possible. The resulting system thus takes the form

$$(A.3) \quad \frac{dW}{dz} = z^r C(z)W = [Q'(z) + z^{-2} B(z)] W$$

where

$$(A.4) \quad z^r C(z) = \left[\begin{pmatrix} q_1'(z) & 0 \\ 0 & q_2'(z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix} \right] + z^{-2} \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix},$$

where $Q'(z)$ in (A.3) is in square brackets in (A.4), $B(z) = [b_{ij}(z)]$ and

$$(A.5) \quad q_i(z) = \sum_{k=0}^r \frac{q_{ik}}{r+1-k} z^{r+1-k} + q_{i,r+1} \log z. \quad (i = 1, 2).$$

If $r \leq -2$ then $Q'(z) = 0$. If for some integer p in $0 \leq p \leq r+1$ $q_{1p} - q_{2p} \neq 0$ then $\xi = 0$. Otherwise ξ may be either zero or one. The functions $b_{ij}(z)$ ($i, j = 1, 2$) have the expansion

$$(A.6) \quad b_{ij}(z) = \sum_{k=0}^{s-1} b_{ijk} z^{-k} + b_{ijs}(z) z^{-s} \sim \sum_{k=0}^{\infty} b_{ijk} z^{-k}$$

where for each positive integer s the elements $b_{ijs}(z)$ ($i, j = 1, 2$) are holomorphic and bounded in a domain $\mathcal{D}$ extending to infinity.

According to the analysis of Chapter II the system (A.3) has a formal solution matrix of the form

$$(A.7) \quad \tilde{W}(z) = \tilde{U}(z) e^{Q(z)}$$

where

$$Q(z) = \begin{bmatrix} q_1(z) & 0 \\ 0 & q_2(z) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \xi \log z & 0 \end{bmatrix}$$

$$(A.8) \quad \tilde{U}(z) \cong \sum_{k=0}^{\infty} U_k z^{-k}; \quad U_0 = I.$$

$$U_k = \begin{bmatrix} u_{11k} & u_{12k} \\ u_{21k} & u_{22k} \end{bmatrix} \quad k = 0, 1, 2, \dots$$

Suppose that there is a smallest integer p in $0 \leq p \leq r+1$ such that $q_{1p} - q_{2p} \neq 0$. Then the U_{ijk} 's are computed from

$$u_{iio} = 1 \quad (i = 1, 2)$$

$$u_{ijk} = 0 \quad \text{if } i \neq j \quad \text{and} \quad 0 \leq k \leq r+1-p \quad (i, j = 1, 2)$$

$$(A.9) \quad u_{iik} = - (1/k) \sum_{s=0}^{k-1} \left\{ b_{i1s} u_{1i, k-1-s} + b_{i2s} u_{2i, k-1-s} \right\} \quad (i = 1, 2; k \geq 1)$$

$$u_{ij, r+2+k-p} = \frac{1}{q_{ip} - q_{jp}} \left[\sum_{s=p+1}^{r+1} \left\{ u_{ij, r+2+k-s} (q_{js} - q_{is}) \right\} \right. \\ \left. + (k+1)u_{ij, k+1} + \sum_{s=0}^{k-1} \left\{ b_{i1s} u_{1j, k-s} + b_{i2s} u_{2j, k-s} \right\} \right]^* \\ (i \neq j; k \geq 1)$$

Note that in this case each column vector can be computed independently of the other.

If on the other hand $q_{1p} - q_{2p} = 0$, $p = 0, 1, \dots, r+1$, then U_{k+1} ($k \geq 0$; $U_0 = I$) is computed from

$$(A.10) \quad (k+1)U_{k+1} + J U_{k+1} - U_{k+1} J + \sum_{s=0}^k B_s U_{k-s} = 0$$

with

$$(A.11) \quad J = \begin{bmatrix} 0 & 0 \\ \xi & 0 \end{bmatrix}; \quad \xi = 0 \quad \text{or} \quad 1.$$

* The sum $\sum_{s=p+1}^{r+1} \{ \quad \}$ is to be replaced by zero if $p = r+1$.

In either case, whether $\xi = 0$ or 1 , with $U_0 = I$ equation (A.10) uniquely determines each U_k , $k \geq 1$. If $\xi = 0$ each column vector can be computed independently of the other. If $\xi = 1$ the second column vector in U_k ($k = 0, 1, \dots$) can be computed without computing the first. On the other hand if $\xi = 1$ and the first column vector in U_k ($k = 0, 1, 2, \dots$) is desired the second column vector must also be computed, and the computations for the second column vector must precede the computations for the first.

We shall obtain an actual vector solution corresponding to the first column vector of formal solutions, and we shall obtain a bound on the difference between this actual solution and a partial sum of a formal solution.

The partial sum

$$(A.12) \quad \varphi_{1m} = \varphi_{1m}(z) = \sum_{k=0}^{m-1} \left\{ \begin{pmatrix} u_{11k} \\ u_{21k} \end{pmatrix} + \begin{pmatrix} u_{12k} \\ u_{22k} \end{pmatrix} \xi \log z \right\} z^{-k} e^{q_1(z)}$$

satisfies the differential equation

$$(A.13) \quad \varphi'_{1m}(z) - z^r C(z) \varphi_{1m}(z) = R_{1m}(z) e^{q_1(z)}$$

where, if $q_1(z) \neq q_2(z)$ (and hence $\xi = 0$) then

$$(A.14) \quad R_{11m}(z) = m u_{11m} z^{-m-1} - \sum_{s=0}^{m-1} [b_{11,m-s}(z) u_{11s} + b_{12,m-s}(z) u_{21s}] z^{-m-2}$$

$$\begin{aligned} R_{21m}(z) &= q'_{21}(z) u_{21m} z^{-m} - \sum_{k=p+1}^{r+1} (q_{2k} - q_{1k}) z^{r-k} u_{21m} z^{-m} \\ &- \sum_{\mu=m+p+1}^{m+r+1} \left[\sum_{s=\mu+1-m}^{\min(\mu, r+1)} (q_{2s} - q_{1s}) u_{21, \mu-s} + (\mu-r-1) u_{21, \mu-r-1} \right. \\ &\quad \left. + \sum_{s=0}^{\mu-r-2} (b_{11s} u_{11, \mu-r-2-s} + b_{12s} u_{21, \mu-r-2-s}) \right] z^{r-\mu} \\ &- \sum_{s=0}^{m-1} [b_{11,m-s}(z) u_{11s} + b_{12,m-s}(z) u_{21s}] z^{-m-2} \end{aligned}$$

where p is the smallest integer such that $q_{2p} - q_{1p} \neq 0$.

On the other hand, if $q_1(z) = q_2(z)$ then

$$(A.15) \quad \begin{pmatrix} R_{11m}(z) \\ R_{21m}(z) \end{pmatrix} = \left\{ \left[- \begin{pmatrix} u_{11m} & u_{12m} \\ u_{21m} & u_{22m} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ \xi & m \end{pmatrix} \begin{pmatrix} u_{11m} & u_{12m} \\ u_{21m} & u_{22m} \end{pmatrix} \right] z^{-m-1} \right. \\ \left. - \sum_{s=0}^{m-1} \begin{pmatrix} b_{11,m-s}(z) & b_{12,m-s}(z) \\ b_{21,m-s}(z) & b_{22,m-s}(z) \end{pmatrix} \begin{pmatrix} u_{11s} & u_{12s} \\ u_{21s} & u_{22s} \end{pmatrix} z^{-m-2} \right] \left\} \begin{pmatrix} 1 \\ \xi \log z \end{pmatrix}.$$

Let

$$(A.16) \quad W_1(z) = \begin{pmatrix} w_{11}(z) \\ w_{21}(z) \end{pmatrix}$$

be an actual solution vector which satisfies the equation (A.3). Then the error vector $\eta_{1m}(z) = w_1(z) - \phi_{1m}(z)$ satisfies the differential equation

$$(A.17) \quad \eta_{1m}(z) - Q'(z)\eta_{1m}(z) = z^{-2} B(z)\eta_{1m}(z) - R_{1m}(z) e^{q_1(z)}.$$

Let us choose a non-negative number a such that if $q_{1p} = q_{2p}$, $p = 0, 1, \dots, r$ then $a = 0$ if $|\zeta| = \infty$ and $a = |\zeta|$ if $|\zeta| < \infty$. If for some $p \in (0, 1, \dots, r)$ $q_{1p} \neq q_{2p}$ then we set $a = a_0$ where $a_0 = \max \{ |u| \mid q'_{21}(u) = 0 \}$ if $|\zeta| = \infty$, while if $|\zeta| < \infty$ we assume $\zeta \in \bar{\mathcal{D}}$, ζ does not coincide with a zero of $q'_{21}(z)$ and $|\zeta| \geq a_0$. Let $\mathcal{D}^* = \bar{\mathcal{D}} - \mathcal{E}_a - S$ where $\mathcal{E}_a$ is the closed disc with center at the origin and radius a , S is empty or $\{\infty\}$ according to whether or not $Q'(z) = 0$. We now define a region $\mathcal{D}(z, \zeta)$ to be the union of all points z ($\neq \zeta$) such that there is a path $\mathcal{P}$ connecting z and ζ which satisfies the following conditions:

- 1) Except for ζ , if $\zeta \in \bar{\mathcal{D}}^* - \mathcal{D}^*$, $\mathcal{P}$ lies entirely in $\mathcal{D}^*$;
- 2) $\mathcal{P}$ consists of a finite number of Jordan arcs $t = t(s)$ such that $\frac{dt(s)}{ds}$ is continuous and non-vanishing;
- 3) $\exp [q_{21}(\tau) - q_{21}(t)]$ is bounded for all points t, τ in the order ζ, t, τ, z on $\mathcal{P}$. If $q_{1p} = q_{2p}$ for $p = 0, 1, \dots, r$ and either $q_{1,r+1} \neq q_{2,r+1}$ or $\xi \neq 0$ we also require that τ/t remain bounded;
- 4) $\gamma_{\mathcal{P}}(t^{\eta-1})$ is bounded, where η may be taken to be zero if $\zeta = 0$ but otherwise η is an arbitrary positive number less than 1.

For purposes of obtaining actual solutions for which the formal solutions are asymptotic expansions, and in order to obtain sharp error bounds it is again desirable (whenever possible) to choose $|\zeta| = \infty$.

If it is possible to choose ζ such that $|\zeta| = \infty$, then by making an appropriate rotary transformation $z = \omega \xi$ with ω a constant and $|\omega| = 1$ it is possible to make $\zeta = \infty$ an end-point of integration. Moreover if the case when coefficient matrix $C(z)$ is holomorphic for all z sufficiently large, the formal solutions of the system (A.3) will then be asymptotic expansions of actual solutions $W_{jm;k}(z)$ (which we shall obtain below) in sectors $|\arg z - \frac{2\pi k}{r+1-p}| \leq \frac{1}{r+1-p} (\frac{3\pi}{2} - \epsilon)$ ($k = 0, 1, \dots, r-p$) where p is the smallest non-negative integer in $0 \leq p \leq r$ such that $q_{1p} \neq q_{2p}$, and ϵ is an arbitrary positive number less than $\frac{3\pi}{4}$; in this last case the corresponding end-points of integration are $\zeta_k = \infty \exp \left[\frac{2\pi \sqrt{-1} k}{r+1-p} \right]$ ($k = 0, 1, \dots, r-p$).

In Appendix B it is shown that if $\mathcal{D}(z, \zeta)$ contains a point distinct from ζ , then $\mathcal{D}(z, \zeta)$ is an open connected domain.

It follows that if $\eta_{1m}(z)$ satisfies the equation

$$(A.18) \quad \eta_{1m}(z) = \int_{\zeta}^z \exp [Q(z) - Q(t)] \left\{ t^{-2} B(t) \eta_{1m}(t) - R_{1m}(t) e^{q_1(t)} \right\} dt$$

where the integral traverses a path $\mathcal{P}$ as described above then $\eta_{1m}(z)$ also satisfies (A.17). On substituting $\eta_{1m}(z) = \epsilon_{1m}(z) e^{q_1(z)}$ in (A.18) this equation may be written

$$(A.19) \quad \epsilon_{1m}(z) = \int_{\zeta}^z \exp [(Q(z) - Iq_1(z)) - (Q(t) - Iq_1(t))] \left\{ t^{-2} B(t) \epsilon_{1m}(t) - R_{1m}(t) \right\} dt$$

Let us now estimate the integral

$$(A.20) \quad R_{1m}^*(z) = - \int_{\xi}^z \exp [(Q(z) - Iq_1(z)) - (Q(t) - Iq_1(t))] R_{1m}(t) dt$$

If for some integer p in $0 \leq p \leq r+1$ we have $q_{2p} \neq q_{1p}$ it is again convenient to define*

$$(A.21) \quad p_{21}(t) = R_{21m}(t) - q_{21}'(t) u_{21m} t^{-m} + \sum_{s=0}^{m-1} [b_{11,m-s}(t) u_{11s} + b_{12,m-s}(t) u_{21s}] t^{m-2}$$

where $R_{21m}(t)$ is given in (A.14). Then, proceeding as in Section 3.4 we find that

$$(A.22) \quad R_{11m}^*(z) = u_{11m} (z^{-m} - \xi^{-m}) + \int_{\xi}^z \sum_{s=0}^{m-1} [b_{11,m-s}(t) u_{11s} + b_{12,m-s}(t) u_{21s}] t^{m-2} dt$$

$$\begin{aligned} R_{21m}^*(z) &= u_{21m} (z^{-m} - e^{q_{21}(z) - q_{21}(\xi)} \xi^{-m}) + \gamma_{21m}^*(z) \\ &+ \int_{\xi}^z e^{q_{21}(z) - q_{21}(t)} \left\{ - \frac{d}{dt} \left(\frac{p_{21}(t)}{q_{21}'(t)} \right) + \sum_{s=0}^{m-1} [b_{21,m-s}(t) u_{11s} + \right. \\ &\left. + b_{22,m-s}(t) u_{21s}] t^{m-2} \right\} dt \end{aligned}$$

where

$$(A.23) \quad \gamma_{21m}^*(z) = \frac{p_{21}(z)}{q_{21}'(z)} - e^{q_{21}(z) - q_{21}(\xi)} \frac{p_{21}(\xi)}{q_{21}'(\xi)}$$

On the other hand, if $q_1(z) = q_2(z)$, then

* Note that $p_{21}(t) \equiv 0$ if the smallest integer p in $0 \leq p \leq r+1$ such that $q_{2p} - q_{1p} \neq 0$ is $r+1$.

$$(A.24) \quad \begin{pmatrix} R_{11m}^*(z) \\ R_{21m}^*(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi \log \frac{z}{t} & 1 \end{pmatrix} \begin{pmatrix} u_{11m} & u_{21m} \\ u_{12m} & u_{22m} \end{pmatrix} \begin{pmatrix} 1 \\ \xi \log t \end{pmatrix} t^{-m} \Big|_{t=\zeta}^{t=z} \\ + \int_{\zeta}^z \begin{pmatrix} 1 & 0 \\ \xi \log \frac{z}{t} & 1 \end{pmatrix} \sum_{s=0}^{m-1} \begin{pmatrix} b_{11,m-s}(t) & b_{12,m-s}(t) \\ b_{21,m-s}(t) & b_{22,m-s}(t) \end{pmatrix} \begin{pmatrix} u_{11s} & u_{12s} \\ u_{21s} & u_{22s} \end{pmatrix} \\ \begin{pmatrix} 1 \\ \xi \log t \end{pmatrix} t^{-m-2} dt$$

We thus define non-negative numbers u_{ilm}^* ($i = 1, 2$) by

$$(A.25) \quad u_{11m}^* = \sup_{\tau \in \mathcal{P}} \left| \frac{(u_{11m} + \xi u_{12m} \log \tau) \tau^{-m} - (u_{11m} + \xi u_{21m} \log \zeta) \zeta^{-m}}{\tau^{\eta_1-m} - \zeta^{\eta_1-m}} \right| \\ u_{21m}^* = \sup_{\tau \in \mathcal{P}} \left| \frac{(u_{21m} + \xi u_{22m} \log \tau) \tau^{-m} e^{q_{21}(\tau) - q_{21}(\zeta)} \{u_{21m} + \xi [\log \frac{\tau}{\zeta} (u_{11m} + u_{21m} \log \zeta) + u_{22m} \log \zeta]\} \zeta^{-m}}{\tau^{\eta_1-m} - \zeta^{\eta_1-m}} \right|$$

where η_1 may be taken zero if ξ is zero, but otherwise η_1 is an arbitrary (fixed) positive number less than 1.

Similarly, with t_1 and t_2 points on $\mathcal{P}$ in the order ζ, t_1, t_2, z we define

$$(A.26) \quad \gamma_{11m} = \sup_{t \in \mathcal{P}} \left| \frac{t^{-\eta_1}}{m+1-\eta_1} \sum_{s=0}^{m-1} \{b_{11,m-s}(t) [u_{11s} + \xi \log t u_{12s}] + \right. \\ \left. + b_{12,m-s}(t) [u_{21s} + \xi \log t u_{22s}]\} \right|$$

$$\begin{aligned}
 \gamma_{21m} = & \sup_{t_1, t_2 \in \mathcal{P}} \left| \frac{\gamma_{21m}^*(t_2)}{t_2^{\eta_1-m} - \xi^{\eta_1-m}} + \right. \\
 & + \frac{t_1^{-\eta_1}}{m+1-\eta_1} \left[e^{q_{21}(t_2)-q_{21}(t_1)} \left\{ -\frac{d}{dt_1} \left(\frac{p_{21}(t_1)}{q'_{21}(t_1)} \right) t_1^{m+2} + \right. \right. \\
 & + \sum_{s=0}^{m-1} \left\{ \left[b_{21,m-s}(t_1) + \xi \log \left(\frac{t_2}{t_1} \right) b_{11,m-s}(t_1) \right] \left[u_{11s} + \xi \log t_1 u_{22s} \right] + \right. \\
 & + \left. \left[b_{22,m-s}(t_1) + \xi \log \left(\frac{t_2}{t_1} \right) b_{12,m-s}(t_1) \right] \left[u_{21s} + \xi \log t_1 u_{22s} \right] \right\} \\
 & \left. \left. + t_1^{m+2} \frac{d}{dt_1} \left(\frac{p_{21}(t_1)}{q'_{21}(t_1)} \right) \right] \right|
 \end{aligned}$$

where it is of course presumed that the terms γ_{21m}^* and $p_{21}(t)/q'_{21}(t)$ are missing if $q_1(t) = q_2(t)$. Conversely if $q_{21}(t) \neq 0$ then ξ is to be taken zero. If $\xi = 0$ then η_1 can be taken to be zero; otherwise η_1 is an arbitrary positive number less than 1.

We also define α , β , γ and δ by

$$\begin{aligned}
 (1-\eta)\alpha &= \sup_{t \in \mathcal{P}} |t^{-\eta} b_{11}(t)|; \quad (1-\eta)\beta = \sup_{t \in \mathcal{P}} |t^{-\eta} b_{12}(t)| \\
 (A.27) \quad (1-\eta)\gamma &= \sup_{t_1, t_2 \in \mathcal{P}} \left| t_1^{-\eta} \left[b_{21}(t_1) + \xi \log \left(\frac{t_2}{t_1} \right) b_{11}(t_1) \right] e^{q_{21}(t_2)-q_{21}(t_1)} \right| \\
 (1-\eta)\delta &= \sup_{t_1, t_2 \in \mathcal{P}} \left| t_1^{-\eta} \left[b_{22}(t_1) + \xi \log \left(\frac{t_2}{t_1} \right) b_{12}(t_1) \right] e^{q_{21}(t_2)-q_{21}(t_1)} \right|
 \end{aligned}$$

where t_1 and t_2 are taken as in (A.26) and η may be taken zero if ξ is zero but otherwise η is an arbitrary (fixed) positive number less than 1.

By the analysis of Appendix B the solution to equation (A.19) exists and is a unique vector of functions holomorphic in $\mathcal{D}(z, \xi)$.

Theorem A.1 If corresponding to the formal vector solution $\tilde{W}_1(z)$ of equation (A.3) we can define a region $\mathcal{D}(z, \xi)$ as described by the conditions on page 123 then the equation (A.3) possesses an actual solution vector

$$(A.28) \quad W_{1m}(z) = \begin{Bmatrix} W_{11m}(z) \\ W_{21m}(z) \end{Bmatrix} = \left\{ \sum_{k=0}^{m-1} \left[\begin{pmatrix} u_{11k} \\ u_{21k} \end{pmatrix} + \xi \log z \begin{pmatrix} u_{12k} \\ u_{22k} \end{pmatrix} \right] z^{-k} + \begin{pmatrix} \epsilon_{11m}(z) \\ \epsilon_{21m}(z) \end{pmatrix} \right\} e^{q_1(z)}$$

of functions holomorphic in $\mathcal{D}(z, \xi)$, such that

$$(A.29) \quad \begin{pmatrix} |\epsilon_{11m}(z)| \\ |\epsilon_{21m}(z)| \end{pmatrix} \leq \exp \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \gamma_{\rho}(t^{\eta-1}) \right] \begin{pmatrix} u_{11m}^* \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{11m} \gamma_{\rho}(t^{\eta_1-m-1}) \\ u_{21m}^* \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{21m} \gamma_{\rho}(t^{\eta_1-m-1}) \end{pmatrix}$$

for all z in $\mathcal{D}(z, \xi)$. The solution $W_{1m}(z)$ depends on ξ and an arbitrary positive integer m . The function $q_1(z)$ is defined as in (A.4); the numbers u_{ilk} are defined by either (A.9) or (A.10). The numbers u_{ilm}^* , γ_{ilm} and $\alpha, \beta, \gamma, \delta$ are defined by (A.25), (A.26) and (A.27) respectively. If $\xi = 0$, both η and η_1 may be taken zero; otherwise η and η_1 are arbitrary positive numbers less than one.

By the analysis of Section 1.2 we can use

$$(A.30) \quad \exp \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left[\frac{\sinh \chi}{\chi} \begin{pmatrix} \frac{\alpha-\delta}{2} & \beta \\ \gamma & \frac{\delta-\alpha}{2} \end{pmatrix} + \begin{pmatrix} \cosh \chi & 0 \\ 0 & \cosh \chi \end{pmatrix} \right]$$

$$\chi = \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}$$

to explicitly evaluate the right hand side of (A.29).

APPENDIX B

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF INTEGRAL EQUATIONS

In this Appendix we establish the existence of solutions of integral equations used in the thesis. Among other works Erdely's [37] development of a real variable theory applicable to single Volterra vector integral equations is probably related most closely to the analysis of this appendix. Generalizations such as the contraction mapping principle (see e.g. [9]) can also be made applicable. We shall however construct a proof along the lines of the usual differential or integral equations proofs which at the same time applies directly to our problem and conditions.

As in [20] we first establish the existence of a domain, we then proceed to construct a sequence of vectors which converges to a solution. In order to avoid unnecessary repetition we shall establish existence and uniqueness only for the general singular system. Two different cases arise however: (a) the case of a single Volterra (vector) integral equation, and (b) the case of a simultaneous pair of Volterra (vector) integral equations. We shall consider each of these cases separately.

B.1 One End-Point of Integration

Lemma B.1 If under the generalized extreme eigenvalue conditions of Section 3.4.2, $\mathcal{D}(z, \zeta)$ contains a point z^* , with z^* bounded and distinct from ζ , then $\mathcal{D}(z, \zeta)$ is an open connected domain.

Proof. $\mathcal{D}^*$ defined on page 85 is certainly a domain. By assumption there is a path $\mathcal{P}$ in $\mathcal{D}^*$ connecting z^* with ζ satisfying the four generalized extreme eigenvalue conditions. Similarly there is such a path $\mathcal{P}$ connecting an arbitrary point z_0 (z_0 bounded) on $\mathcal{P}$ with ζ . We can clearly choose a radius $\rho > 0$ sufficiently small such that when we join an arbitrary point z_1 in the interior of the disc $S(\rho)$ ($S(\rho)$ is entirely in $\mathcal{D}^*$) of radius ρ about z_0 with z_0 by a straight line $\mathcal{P}_1$, then conditions 1), 2) and 4) are satisfied on $\mathcal{P} \cup \mathcal{P}_1$. It remains to be shown that condition 3) is also satisfied.

To achieve this we define matrices

$$K_i(\mathcal{P}) = \sup_{\tau, t \in \mathcal{P}} |\exp[Q_i(\tau) - Q_i(t) - I_{\mu_i}(q_j(\tau) - q_j(t))]| \quad (i=1,2,\dots, \ell; i \neq j) \quad (B.1)$$

$$K_i'(\mathcal{P}_1) = \sup_{\tau_1, t_1 \in \mathcal{P}_1} |\exp[Q_i(\tau_1) - Q_i(t_1) - I_{\mu_i}(q_j(\tau_1) - q_j(t_1))]|$$

of non-negative elements where, in equation (B.1), τ, t, τ_1, t_1 are points on $\mathcal{P} \cup \mathcal{P}_1$ in the order $z_1, \tau_1, t_1, z_0, \tau, t, \zeta$ and where the limit superiors are taken for each individual element of the matrices.

Since the element of the matrices $Q_i(t)$ ($i = 1, 2, \dots, \ell$) are holomorphic functions bounded in $\mathcal{D}$, each of the matrices consists of non-negative bounded elements. In fact the elements of $K_i(\mathcal{P})$ ($i = 1, 2, \dots, \ell; i \neq j$) are bounded by hypothesis of this proof. We now choose τ and t on $\mathcal{P} \cup \mathcal{P}_1$, in the order z_1, τ, t, ζ . We then consider

$$(B.2) \quad K_i(\mathcal{P} \cup \mathcal{P}_1) = \sup_{\tau, t \in \mathcal{P} \cup \mathcal{P}_1} |\exp[Q_i(\tau) - Q_i(t) - I_{\mu_i}(q_j(\tau) - q_j(t))]|. \quad (i = 1, 2, \dots, \ell; i \neq j)$$

If τ (in B.2)) is on $\mathcal{P}$ so is t , and in this case, by hypothesis the matrix on the right consists of bounded elements.

On the other hand if both τ and t are on $\mathcal{P}_1$, the matrix on the right is similarly bounded ($K_i'(\mathcal{P}_1)$ of equation (B.1)). Suppose then, that τ is on $\mathcal{P}_1$ and t is on $\mathcal{P}$. In this case

$$\begin{aligned}
 (B.3) \quad & |\exp [Q_i(\tau)-Q_i(t)-I_{\mu_i}(q_j(\tau)-q_j(t))]| \\
 &= |\exp[Q_i(z_0)-Q_i(t)-I_{\mu_i}(q_j(z_0)-q_j(t))]| \exp[Q_i(\tau)-Q_i(z_0)-I_{\mu_i}(q_j(\tau)-q_j(z_0))]| \\
 &\leq |\exp[Q_i(z_0)-Q_i(t)-I_{\mu_i}(q_j(z_0)-q_j(t))]| |\exp[Q_i(\tau)-Q_i(z_0)-I_{\mu_i}(q_j(\tau)-q_j(z_0))]|
 \end{aligned}$$

so that $K_i(\mathcal{P} \cup \mathcal{P}_1) \leq K_i(\mathcal{P}) K'_i(\mathcal{P}_1)$, which shows that the elements of $K_i(\mathcal{P} \cup \mathcal{P}_1)$ ($i = 1, 2, \dots, l$; $i \neq j$) are bounded, and simultaneously establishes that $\mathcal{D}(z, \zeta)$ is an open connected domain.

We next establish the existence and uniqueness of the solution of the integral equation (3.4.22). To achieve this goal, we first observe that all the known functions (i.e. all functions except the solution vector) in (3.4.22) are holomorphic and single-valued in every simply connected subset of $\mathcal{D}(z, \zeta)$. If $\mathcal{D}(z, \zeta)$ is multiply-connected, we interpret $\mathcal{D}(z, \zeta)$ (or any open subset of $\mathcal{D}(z, \zeta)$) to be a Riemann surface on which all known functions of equation (3.4.22) are holomorphic and single-valued.

We need only establish existence and uniqueness of the solution of (3.4.22) for each τ in an arbitrary open bounded subset $\mathcal{D}^*(z, \zeta)$ of $\mathcal{D}(z, \zeta)$ for, from this, existence and uniqueness follows for all of $\mathcal{D}(z, \zeta)$. We assume ζ is in $\overline{\mathcal{D}^*(z, \zeta)}$ if ζ is bounded. If ζ is unbounded we choose an arbitrary subfamily $\{\mathcal{P}^*\}$ of the family $\{\mathcal{P}\}$ (see Section 3.4.4) such that if $\mathcal{P}$ is in $\{\mathcal{P}^*\}$ then $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$; here $\mathcal{P}'$ is a fixed path from ζ to a boundary point ζ' of $\mathcal{D}^*(z, \zeta)$, while $\mathcal{P}''$ lies in $\mathcal{D}^*(z, \zeta)$ and joins ζ' with an arbitrary point τ of $\mathcal{D}^*(z, \zeta)$. With η defined as

in the definition of the extreme eigenvalue case (Section 3.4.2) we assume in any case — i.e. whether ζ bounded or unbounded — that $\gamma_{\mathcal{P}}(t^{\eta-1})$ is to remain uniformly bounded.[†]

As is usual procedure, we define a sequence $\{h_k(\tau)\}$ ($k = 0, 1, \dots$) of $n \times 1$ vectors of holomorphic functions by (see equation (3.4.22))

$$h_0(\tau) = 0$$

(B.4)

$$h_{k+1}(\tau) = \int_{\zeta}^{\tau} e^{D_j(\tau) - D_j(t)} [t^{-2} B(t) h_k(t) - \sum_{s=\nu}^{\mu} R_{jm}^{\nu}(t) \frac{(\log t)^{s-\nu}}{(s-\nu)!}] dt$$

where τ is any point on $\mathcal{P}$. The integral is taken over $\mathcal{P}$ where $\mathcal{P}$ satisfies the generalized extreme eigenvalue conditions in Section 3.4.2, together with the remarks above.

Let us next show that the sequence $\{h_k(\tau)\}$ is well defined, and that each element of this sequence of vectors is bounded. By equation (B.4) we have

$$h_1(\tau) = - \int_{\zeta}^{\tau} e^{D_j(\tau) - D_j(t)} \sum_{s=\nu}^{\mu} R_{jm}^{\nu}(t) \frac{(\log t)^{s-\nu}}{(s-\nu)!} dt$$

(B.5)

$$h_{k+1}(\tau) - h_k(\tau) = \int_{\zeta}^{\tau} e^{D_j(\tau) - D_j(t)} t^{-2} B(t) [h_k(t) - h_{k-1}(t)] dt \quad (k = 1, 2, \dots).$$

[†] In our proof of existence and uniqueness all our contour integrals will be "independent of the path", and hence there is no loss of generality in the restriction from $\{\mathcal{P}\}$ such that $\gamma_{\mathcal{P}}(t^{\eta-1})$ is bounded to $\{\mathcal{P}^*\} \subset \{\mathcal{P}\}$ such that $\gamma_{\mathcal{P}^*}(t^{\eta-1})$ is uniformly bounded.

Clearly $h_1(\tau)$ is a well defined vector of holomorphic functions of τ , for each τ in $\mathcal{D}^*(z, \zeta)$. Further

$$(B.6) \quad |h_1(\tau)| \leq U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})$$

where the vector on the right of (B.6) is defined as in Theorem 3.4.1.

Suppose that we have verified existence and holomorphy for $h_k(\tau)$,

$k \geq 1$, together with the inequalities

$$(B.7) \quad |h_k(\tau) - h_{k-1}(\tau)| \leq \frac{\{B \gamma_{\zeta, \tau}(t^{\eta-1})\}^{k-1}}{(k-1)!} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\}$$

where B and η_1 are defined as in Theorem 3.4.1. Adding the inequalities we get

$$\begin{aligned} (B.8) \quad |h_k(\tau)| &\leq \sum_{s=1}^k |h_s(\tau) - h_{s-1}(\tau)| \\ &\leq \sum_{s=1}^k \frac{\{B \gamma_{\zeta, \tau}(t^{\eta-1})\}^{s-1}}{(s-1)!} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\} \\ &< \exp \{B \gamma_{\zeta, \tau}(t^{\eta-1})\} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\} \end{aligned}$$

This implies by (B.4) that $h_{k+1}(\tau)$ is also well defined as a vector of holomorphic functions of τ . Further, from the second of (B.5),

(3.4.23) and (B.7) we get

$$(B.9) \quad |h_{k+1}(\tau) - h_k(\tau)|$$

$$\begin{aligned} &\leq \int_{\xi}^{\tau} |(1-\eta)Bt^{\eta-2}| \frac{\{B \xi, t(t^{\eta-1})\}^{k-1}}{(k-1)!} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\} |dt| \\ &\leq \int_{\xi}^{\tau} \frac{\{B \xi, t(t^{\eta-1})\}^{k-1}}{(k-1)!} d\{B \gamma_{\xi, t}(t^{\eta-1})\} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\} \\ &= \frac{\{B \gamma_{\xi, \tau}(t^{\eta-1})\}^k}{k!} \{U_{jm}^{\nu*} \gamma_{\rho}(t^{\eta_1-m}) + \gamma_{jm}^{\nu} \gamma_{\rho}(t^{\eta_1-m-1})\} \end{aligned}$$

so that (B.7) holds for all k . The same is true for (B.8).

Thus we see that the sequence $\{h_k(\tau)\}$ is well defined for τ in $\mathcal{D}^*(z, \xi)$. Further (B.7) implies that (see e.g. [18] pages 166-167)

$$(B.10) \quad W_{jm}^{\nu}(\tau) \equiv \lim_{k \rightarrow \infty} h_k(\tau)$$

exists uniformly with respect to τ in $\mathcal{D}^*(z, \xi)$. Thus $W_{jm}^{\nu}(\tau)$ is holomorphic in this domain. Moreover from (B.4) and the uniform convergence it follows that $W_{jm}^{\nu}(z)$ satisfies (3.4.22), and consequently it is a desired solution of this equation.

It remains to show uniqueness. Suppose $W(z) \neq W_{jm}^{\nu}(z)$ satisfies (3.4.22), where the elements of $W(z)$ are uniformly bounded on a path ρ described above. Then

$$(B.11) \quad W(\tau) - W_{jm}^{\nu}(\tau) = \int_{\xi}^{\tau} e^{D_j(\tau) - D_j(t)} B(t) t^{-2} \{W(t) - W_{jm}^{\nu}(t)\} dt$$

and hence, setting $\Delta(\tau) = \sup_{t \in \rho} |W(t) - W_{jm}^{\nu}(t)|$, iterating (B.11)

and using (3.4.23) we get

$$(B.12) \quad |W(\tau) - W_{jm}^v(\tau)| \leq \frac{\{B \Upsilon_{\zeta, \tau}(t^{\eta-1})\}^k}{k!} \Delta(\tau)$$

where k is an arbitrary positive integer. However, the right hand side of (B.12) tends to the zero vector as $k \rightarrow \infty$, so that the assumption $W(z) \neq W_{jm}^v(z)$ is untenable. This proves that the solution (B.10) of equation (3.4.22) is unique in the space of $n \times 1$ vectors of functions holomorphic in $\mathcal{O}(z, \zeta)$ which remain bounded as $z \rightarrow \zeta$ along any path $\mathcal{P}$ specified above.

Theorem B.1 The solution to equation (3.4.22) exists and is a unique vector of functions holomorphic on the Riemann surface $\mathcal{O}(z, \zeta)$ defined by the extreme eigenvalue conditions in Section 3.4.2. Moreover, the inequality (B.8) constitutes a bound for this solution.

B.2 Two End-Points of Integration

Lemma B.2 If under the generalized interior eigenvalue conditions of Section 3.4.2 (page 86), $\mathcal{O}(\zeta_1, \zeta_2, \eta)$ contains z^* with z^* bounded and distinct from ζ_1 and ζ_2 , then $\mathcal{O}(\zeta_1, \zeta_2, \eta)$ is an open connected domain.

Proof. By assumption there is a path $\mathcal{P}$ in $\mathcal{O}^*$ ($\mathcal{O}^*$ is defined on page 85) connecting ζ_1 and ζ_2 , passing through z^* and satisfying the four generalized interior eigenvalue conditions of Section 3.4.2. The same path $\mathcal{P}$ connects any arbitrary point $z_0 \in \mathcal{P}$ in this manner. By an argument similar to that used in the proof of Lemma B.1 we can clearly choose a radius $\rho > 0$ sufficiently small such that when we alter $\mathcal{P}$ to pass

through the point z_1 instead of through z_0 as indicated in Figure 3, where z_1 is an arbitrary point interior to the disc $S(\rho)$ then conditions 1), 2) and 4) of the generalized interior eigenvalue conditions remain satis-

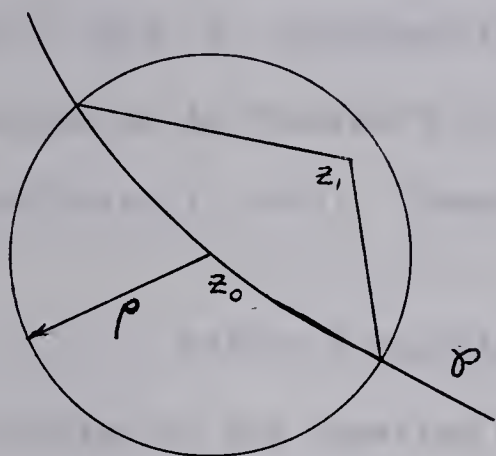


FIGURE 3

fied. Condition 3) can however also be made satisfied by taking ρ sufficiently small. To prove this we need merely show that as $\rho \rightarrow 0$ and the altered path ρ' in Figure 3 tends to ρ ,

$$(B.13) \quad \sup_{t, \tau \in \rho} \|e^{D_{j1}(\tau) - D_{j1}(t)} \alpha(t) t^{-\eta}\| \rightarrow \sup_{t, \tau \in \rho} \|e^{D_{j1}(\tau) - D_{j1}(t)} \alpha(t) t^{-\eta}\|$$

where t and τ are points located on the paths ρ and ρ' in the order $\zeta_1, t, \tau, \zeta_2$. A more convenient way of achieving this is to put

$$(B.14) \quad \Delta(t, t_1, \tau, \tau_1) = e^{D_{j2}(\tau_1) - D_{j2}(t_1)} \alpha(t_1) t_1^{-\eta} - e^{D_{j2}(\tau) - D_{j2}(t)} \alpha(t) t^{-\eta}$$

where t and τ are located as described above, and $t_1 = t$, $\tau_1 = \tau$ outside $S(\rho)$ while in the interior of $S(\rho)$, t_1 and τ_1 are arbitrarily located on ρ . Let us denote the norm on the left in (B.13) by α' and that on the right by α . Then it follows that

$$(B.15) \quad \alpha' \leq \alpha + \sup_{t_1, \tau_1 \in \rho'; t, \tau \in \rho} \|\Delta(t, t_1, \tau, \tau_1)\|.$$

The norm on the right of (B.15) is defined as a sum of absolute values of holomorphic functions. With $f(z)$ holomorphic, $\log |f(z)|$ is harmonic; the maximum modulus principle (see Whitney [3]) applies and the norm on the right of (B.15) tends to zero as $\rho \rightarrow 0$. A similar argument applies to β , γ and δ . Moreover the difference $|\gamma_{\rho}(t^{\eta-1}) - \gamma_{\rho}(t^{\eta-1})|$ (with η chosen as in Theorem 3.4.3) can be made as small as we please by taking ρ sufficiently small. Lemma B.2 follows.

Before proceeding to establish existence and uniqueness of the solution of the equation (3.4.28) we again observe that if $\mathcal{D}(\xi_1, \xi_2, \eta)$ is simply connected then all known functions of equation (3.4.28) are single-valued on $\mathcal{D}(\xi_1, \xi_2, \eta)$. If on the other hand $\mathcal{D}(\xi_1, \xi_2, \eta)$ is multiply-connected then we interpret $\mathcal{D}(\xi_1, \xi_2, \eta)$ (or any open subset of $\mathcal{D}(\xi_1, \xi_2, \eta)$) to be a Riemann surface on which all known functions of equation (3.4.28) are holomorphic and single-valued.

We again need only establish existence and uniqueness of the solution of (3.4.28) for each τ in an arbitrary open bounded subset $\mathcal{D}^*(\xi_1, \xi_2, \eta)$ of $\mathcal{D}(\xi_1, \xi_2, \eta)$, for from this, existence and uniqueness follows for all of $\mathcal{D}(\xi_1, \xi_2, \eta)$.

Let us choose an arbitrary but fixed path $\mathcal{P}$ belonging to the family of paths $\{\mathcal{P}\}$ which satisfies the conditions of Section 3.4.2. We next pick any two distinct points ξ_1' and ξ_2' located on $\mathcal{P}$ in the order $\xi_1, \xi_1', \xi_2', \xi_2$. Further, we assume that ξ_1' and ξ_2' are bounded. Assume moreover that with α , β , γ and δ defined as for Condition 4) (Interior eigenvalue conditions — Section 3.4.2) we have $\frac{1}{2}\{\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\}\gamma_{\rho}(t^{\eta-1}) = 1 - \epsilon$

where ϵ is a positive number less than 1. We now form a class $\{\mathcal{P}^*\} \subset \{\mathcal{P}\}$ of paths such that each $\mathcal{P}^*$ coincides with the fixed path $\mathcal{P}$ everywhere except on that part of $\mathcal{P}$ which joins ζ_1' and ζ_2' ; here we allow $\mathcal{P}^*$ to vary such that $\frac{1}{2}\{\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\} \eta^{\eta-1} < 1 - \frac{1}{2}\epsilon$. Thus we define $\mathcal{O}^*(\zeta_1, \zeta_2, \eta) = \bigcup \mathcal{P}_1^*$ where $\mathcal{P}_1^*$ is that part of $\mathcal{P}^*$ which is left after we remove the portions from ζ_1 to ζ_1' and from ζ_2' to ζ_2 .

We again define a sequence of $n \times 1$ vectors of holomorphic functions

$$\{h_k(\tau)\} = \begin{Bmatrix} h_{k1}(\tau) \\ h_{k2}(\tau) \end{Bmatrix} \quad (k = 0, 1, 2, \dots)$$

by

$$h_0(\tau) = \begin{pmatrix} h_{01}(\tau) \\ h_{02}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(B.16)

$$\begin{pmatrix} h_{k1}(\tau) \\ h_{k2}(\tau) \end{pmatrix} = \begin{pmatrix} \int_{\zeta_1}^{\tau} 0 \\ 0 \int_{\zeta_2}^{\tau} \end{pmatrix} \begin{pmatrix} e^{D_{j1}(\tau) - D_{j2}(t)} & 0 \\ 0 & e^{D_{j2}(\tau) - D_{j2}(t)} \end{pmatrix} t^{-2}$$

$$\begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \begin{pmatrix} h_{k-1,1}(t) \\ h_{k-1,2}(t) \end{pmatrix} dt + \begin{pmatrix} R_{jm1}^*(\tau) \\ R_{jm2}^*(\tau) \end{pmatrix} e^{-q_j(\tau)} \quad (k \geq 1).$$

where τ is any point on $\mathcal{P}^*$, where the first integral is over a path $\mathcal{P}^*$ from ζ_1 to τ and the second is over $\mathcal{P}^*$ from τ to ζ_2 .

The sequence $\begin{Bmatrix} h_{k1}(\tau) \\ h_{k2}(\tau) \end{Bmatrix} \quad (k = 0, 1, \dots)$ is well defined.

For, by equation (B.16) we have

$$(B.17) \quad \begin{pmatrix} h_{11}(\tau) \\ h_{12}(\tau) \end{pmatrix} = \begin{pmatrix} R_{jm1}^*(\tau) \\ R_{jm2}^*(\tau) \end{pmatrix} e^{-q_j(\tau)}$$

$$\begin{pmatrix} h_{k+1,1}(\tau) - h_{k1}(\tau) \\ h_{k+1,2}(\tau) - h_{k2}(\tau) \end{pmatrix} = \begin{pmatrix} \int_{\xi_1}^{\tau} 0 \\ 0 \int_{\xi_2}^{\tau} \end{pmatrix} \begin{pmatrix} e^{D_{j1}(\tau) - D_{j1}(t)} & 0 \\ 0 & e^{D_{j2}(\tau) - D_{j2}(t)} \end{pmatrix} \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} h_{k1}(t) - h_{k-1,1}(t) \\ h_{k2}(t) - h_{k-1,2}(t) \end{pmatrix} \frac{dt}{t^2} ;$$

by the definition of the vectors $R_{mjk}(\tau)$ ($k = 1, 2$), $h_1(\tau)$ is clearly a well-defined vector of functions holomorphic in $\mathcal{O}^*(\xi_1, \xi_2, \eta)$. Further, if we define functions $\psi_{mk}(\cdot, \cdot)$ ($k = 1, 2$) by equation (3.4.30) then

$$(B.18) \quad \begin{pmatrix} \|h_{11}(\tau)\| \\ \|h_{12}(\tau)\| \end{pmatrix} \leq \begin{pmatrix} \psi_{m1}(\xi_1, \tau) \\ \psi_{m2}(\tau, \xi_2) \end{pmatrix}.$$

Suppose that we have verified existence and holomorphy for $h_k(\tau)$, $k \geq 1$, together with the inequalities (B.18) and

$$(B.19) \quad \begin{pmatrix} \|h_{k+1,1}(\tau) - h_{k,1}(\tau)\| \\ \|h_{k+1,2}(\tau) - h_{k,2}(\tau)\| \end{pmatrix} \leq \begin{pmatrix} \gamma_{\xi_1, \tau}(t^{\eta-1}) & 0 \\ 0 & \gamma_{\tau, \xi_2}(t^{\eta-1}) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^k \gamma_{\mathcal{O}^*}(t^{\eta-1}) \begin{pmatrix} \psi_{m1}(\xi_1, \xi_2) \\ \psi_{m2}(\xi_1, \xi_2) \end{pmatrix}$$

($k \geq 1$)

This implies by the second of (B.16) that $h_{k+2}(\tau)$ is also well defined as a vector of holomorphic functions of τ , for each τ in $\mathcal{D}^*(\xi_1, \xi_2, \eta)$.

Let us prove the inequality (B.19) for $k = 1$. By equations (B.18), (B.17) and (3.4.31) we find that

$$(B.20) \quad \begin{pmatrix} \|h_{2,1}(\tau) - h_{1,1}(\tau)\| \\ \|h_{2,2}(\tau) - h_{1,2}(\tau)\| \end{pmatrix} \leq \begin{pmatrix} [\alpha\psi_{m1}(\xi_1, \tau) + \beta\psi_{m2}(\xi_1, \xi_2)]\gamma_{\xi_1, \tau}(t^{\eta-1}) \\ [\gamma\psi_{m1}(\xi_1, \xi_2) + \delta\psi_{m2}(\tau, \xi_2)]\gamma_{\tau, \xi_2}(t^{\eta-1}) \end{pmatrix}$$

so that (B.20) is true for $k = 1$. Assuming (B.20) true for some fixed positive integer k , we substitute this inequality into the second of (B.17) to obtain

$$(B.21) \quad \begin{pmatrix} \|h_{k+2,1}(\tau) - h_{k+1,1}(\tau)\| \\ \|h_{k+2,2}(\tau) - h_{k+1,2}(\tau)\| \end{pmatrix} \leq \begin{pmatrix} \int_{\xi_1}^{\tau} 0 \\ 0 \int_{\tau}^{\xi_2} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \|h_{k+1,1}(t) - h_{k,1}(t)\| \\ \|h_{k+1,2}(t) - h_{k,2}(t)\| \end{pmatrix} \left| \frac{(1-\eta)dt}{t^{\eta+2}} \right|$$

$$\leq \begin{pmatrix} \int_{\xi_1}^{\tau} 0 \\ 0 \int_{\tau}^{\xi_2} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{k+1} \gamma_{\mathcal{D}^*}^k(t^{\eta-1}) \begin{pmatrix} \psi_{m1}(\xi_1, \xi_2) \\ \psi_{m2}(\xi_1, \xi_2) \end{pmatrix} \left| \frac{(1-\eta)dt}{t^{\eta+2}} \right|$$

$$= \begin{pmatrix} \gamma_{\xi_1, \tau}(t^{\eta-1}) & 0 \\ 0 & \gamma_{\tau, \xi_2}(t^{\eta-1}) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{k+1} \gamma_{\mathcal{D}^*}^k(t^{\eta-1}) \begin{pmatrix} \psi_{m1}(\xi_1, \xi_2) \\ \psi_{m2}(\xi_1, \xi_2) \end{pmatrix}$$

The first of the above inequalities is obtained directly from the second of (B.17) together with (3.4.31); the second of (B.21) is obtained using (B.20). Thus (B.19) holds for each integer $k \geq 1$. Moreover summing (B.19) and (B.18) we find that under the conditions imposed on $\mathcal{D}^*$,

$$(B.22) \quad \begin{pmatrix} h_{k,1}(\tau) \\ h_{k,2}(\tau) \end{pmatrix} = \begin{pmatrix} \psi_{m1}(\xi_1, \tau) \\ \psi_{m2}(\tau, \xi_2) \end{pmatrix} + \begin{pmatrix} \gamma_{\xi_1, \tau}(t^{\eta-1}) & 0 \\ 0 & \gamma_{\tau, \xi_2}(t^{\eta-1}) \end{pmatrix} E f \left(E \gamma_{\rho}(t^{\eta-1}) \right) \begin{pmatrix} \psi_{m1}(\xi_1, \xi_2) \\ \psi_{m2}(\xi_1, \xi_2) \end{pmatrix}$$

for all integers $k \geq 2$, where the matrices E and $f(\quad)$ are defined as in Theorem 3.4.3. Consequently (see e.g. [18] pages 166-67)

$$(B.23) \quad \begin{pmatrix} W_{jm1}^v(\tau) \\ W_{jm2}^v(\tau) \end{pmatrix} \equiv \lim_{k \rightarrow \infty} \begin{pmatrix} h_{k1}(\tau) \\ h_{k2}(\tau) \end{pmatrix}$$

exists uniformly with respect to τ in $\mathcal{O}^*(\xi_1, \xi_2, \eta)$. Thus $W_{jm}^v(\tau)$ is holomorphic in this domain. Moreover from (B.16) and the uniform convergence it follows that $W_{jm}^v(\tau)$ satisfies (3.4.28), and consequently it is a desired solution of this equation.

In order to show uniqueness we suppose that there exist vectors $W_1(z)$ and $W_2(z)$ of some dimension as $W_{jm1}^v(z)$ and $W_{jm2}^v(z)$ respectively, such that the elements of these vectors are holomorphic in $\mathcal{O}^*(\xi_1, \xi_2, \eta)$, bounded on $\mathcal{P}^*$, and such that

$$(B.24) \quad \begin{pmatrix} W_1(z) \\ W_2(z) \end{pmatrix} \neq \begin{pmatrix} W_{jm1}^v(z) \\ W_{jm2}^v(z) \end{pmatrix}$$

also satisfies (3.4.28). Then

$$(B.25) \quad \begin{pmatrix} W_1(\tau) - W_{jm1}^v(\tau) \\ W_2(\tau) - W_{jm2}^v(\tau) \end{pmatrix} = \begin{pmatrix} \int_{\xi_1}^{\tau} & 0 \\ 0 & \int_{\xi_2}^{\tau} \end{pmatrix} \begin{pmatrix} e^{D_{j1}(\tau) - D_{j1}(t)} & 0 \\ 0 & e^{D_{j2}(\tau) - D_{j2}(t)} \end{pmatrix} \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$$

$$\begin{pmatrix} W_1(t) - W_{jm1}^v(t) \\ W_2(t) - W_{jm2}^v(t) \end{pmatrix} \frac{dt}{t^2}$$

Setting

$$(B.26) \quad \begin{pmatrix} \Delta_1(\tau) \\ \Delta_2(\tau) \end{pmatrix} \equiv \begin{pmatrix} \sup_{\tau \in \mathcal{P}} \|W_1(\tau) - W_{jm1}^v(\tau)\| \\ \sup_{\tau \in \mathcal{P}} \|W_2(\tau) - W_{jm2}^v(\tau)\| \end{pmatrix}$$

and iterating (B.25) we obtain

$$(B.27) \quad \begin{pmatrix} \|W_1(\tau) - W_{jm1}^v(\tau)\| \\ \|W_2(\tau) - W_{jm2}^v(\tau)\| \end{pmatrix} \leq \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \gamma_{\mathcal{P}}(t^{\eta-1}) \right]^k \begin{pmatrix} \Delta_1(\tau) \\ \Delta_2(\tau) \end{pmatrix}$$

where k is an arbitrary positive integer. However the right hand side of (B.27) tends to the zero vector as $k \rightarrow \infty$, so that the assumption (B.24) is absurd. This proves that the solution of the equation (3.4.28) is unique in the class of $n \times 1$ vectors of functions holomorphic in $\mathcal{O}(\xi_1, \xi_2, \eta)$ which remain bounded on any path $\mathcal{P}$ specified above.

Theorem B.2 The solution of equation (3.4.28) exists and is a unique vector of functions holomorphic on the Riemann surface defined by the interior eigenvalue condition in Section 3.4.2. Moreover, the inequality (B.22) constitute a norm bound for this solution.

It is noteworthy that once a holomorphic solution vector is known at some point τ , this same solution vector can be uniquely extended to every bounded subset of $\mathcal{O}$ by a method in [19]. We shall not prove this statement here, since in this thesis we have not exploited this possibility with respect to obtaining sharper error bounds near the origin.

APPENDIX C

SOLUTION OF INTEGRAL INEQUALITIES AND PROOFS OF THEOREMS IN SECTION 3.4.5

In this appendix we state and prove some extensions of Gronwall's [17] (or Bellman's [2]) inequality; we then use these extensions to carry out the proofs of Theorems in Sections 3.3.5 and 3.4.5. For extensions of this inequality to finite difference equations and to (scalar) Volterra integral inequalities for which the kernel may be more general see the papers [43,44].

C.1 Extensions to Matrices

Lemma C.1 For k a positive integer, let φ , G and H be $k \times 1$ vectors and let E and F be $k \times k$ matrices of piecewise continuous non-negative functions of σ in $a \leq \sigma \leq b$ such that $F(\sigma) E(\sigma)$ and $\int_a^\sigma F(s)E(s)ds$ commute. Then if for each σ in $a \leq \sigma \leq b$

$$(C.1) \quad \varphi(\sigma) \leq E(\sigma) \int_a^\sigma [F(s) \varphi(s) + G(s)]ds + H(\sigma)$$

the following inequality also holds in $a \leq \sigma \leq b$:

$$(C.2) \quad \varphi(\sigma) \leq H(\sigma) + E(\sigma) \int_a^\sigma \exp \left\{ \int_s^\sigma F(\tau) E(\tau)d\tau \right\} \{G(s) + F(s)H(s)\}ds$$

Proof. From the statement of the Lemma there exists a $k \times 1$ vector $\psi(\sigma)$ of non-negative piecewise continuous functions such that we can replace the inequality (C.1) by the equality, i.e.

$$(C.3) \quad \varphi(\sigma) = E(\sigma) \int_a^\sigma [F(s) \varphi(s) + G(s)]ds + H(\sigma) - \psi(\sigma).$$

The equation (C.3) is however a Volterra integral equation in φ with

unique* solution (see e.g. [16]).

$$(C.4) \quad \varphi(\sigma) = H(\sigma) - \psi(\sigma) + E(\sigma) \int_a^\sigma \exp\left\{\int_s^\sigma F(\tau)E(\tau)d\tau\right\} \{G(s) + F(s)[H(s) - \psi(s)]\} ds$$

Decreasing the magnitude of the elements $\psi(\sigma)$ increases the magnitude of the elements on the right hand side of (C.4). The inequality (C.2) follows.

Lemma C.2 Let $E^{-1}(\sigma)$ exist for σ in $[a, b]$ and let $\bigvee_{a, b} (E^{-1}H)$ exist. Then along with (C.2) the (less sharp but easier to evaluate) inequality

$$(C.5) \quad \varphi(\sigma) \leq E(\sigma) \exp \left\{ \int_a^\sigma F(\tau)E(\tau)d\tau \right\} [E^{-1}(a)H(a) + \bigvee_{a, \sigma} (E^{-1}H) + \int_a^\sigma G(s)ds]$$

also holds.

Proof. The inequality (C.5) follows directly if we integrate the right of (C.2) by parts.

It is noteworthy that if $\int_{\Delta} dE^{-1}(s) H(s) \geq 0$ over each arbitrary sub-interval Δ on $[a, b]$, then $\bigvee_{a, \sigma} (E^{-1}H) = E^{-1}(\sigma)H(\sigma) - E^{-1}(a)H(a)$; in this case the inequality (C.5) is further simplified. If in addition $E(s)$ is a scalar then (C.5) reduces to

$$(C.6) \quad \varphi(\sigma) \leq \exp \left\{ \int_a^\sigma F(\tau) E(\tau)d\tau \right\} \{H(\sigma) + E(\sigma) \int_a^\sigma G(s)ds\}$$

In this thesis we have not attempted to make full use of this property,

* Unique in the class of $k \times 1$ vectors of piecewise continuous functions.

particularly with respect to our analysis concerning the case of the regular singular point. The derivation of the inequality (C.6) leads us to conjecture that it is possible (by making the transformation $\epsilon_{jm}^{\nu}(z) = z^k \eta_{jm}^{\nu}(z)$ in the integral equation (3.2.9) and appropriately altering the conditions on the paths of integration) to obtain error bounds for the regular singular (and the general) case without modifying the usual canonical forms.

C.2 Extensions to Fredholm Integral Equations

Lemma C.3 For k a positive integer, let G ($k \times k$) H ($k \times k$), φ ($k \times 1$) and θ ($k \times 1$) be matrices of non-negative piecewise continuous functions of s in (a,b) . If in addition we are given that the inequality

$$(C.7) \quad \varphi(\sigma) \leq \int_a^b G(\sigma) H(s) \varphi(s) ds + \theta(\sigma)$$

holds for each σ in (a,b) , and that the eigenvalues of the matrix

$$(C.8) \quad F = \int_a^b G(s) H(s) ds$$

are less than one in magnitude, then the inequality

$$(C.9) \quad \varphi(\sigma) \leq \theta(\sigma) + G(\sigma) \{I - F\}^{-1} \int_a^b H(s) \theta(s) ds$$

also holds.

Proof. From the statement of the lemma there exists a unique $k \times 1$ vector $\psi(\sigma)$ of non-negative piecewise continuous functions such that we can replace the inequality (C.7) by the equality

$$(C.10) \quad \varphi(\sigma) = \int_a^b G(\sigma) H(s) \varphi(s) ds + \theta(\sigma) - \psi(\sigma)$$

The equation (C.10) is however a Fredholm integral equation in φ ; if no eigenvalue of the matrix F (in (C.8)) is one, then (C.10) has the unique* solution (see e.g. [16])

$$(C.11) \quad \varphi(\sigma) = \theta(\sigma) - \psi(\sigma) + G(\sigma) \{I - F\}^{-1} \int_a^b H(s) \{\theta(s) - \psi(s)\} ds$$

If in addition all the eigenvalues of F are less than one in magnitude, then

$$(C.12) \quad \{I - F\}^{-1} = \sum_{v=0}^{\infty} F^v$$

and the sum of the power series on the right of (C.12) is a matrix of non-negative elements. Thus decreasing the magnitude of the elements $\psi(\sigma)$ (and $\psi(s)$) on the right of (C.11) increases the elements of $\varphi(\sigma)$. The inequality (C.9) follows.

C.3 Proofs of Theorems in Sections 3.3.5 and 3.4.5

For sake of avoiding unnecessary repetition we shall only prove the theorems in Section 3.4.5, since the Theorems in Section 3.3.5 are special cases of those in Section 3.4.5

* Unique in the class of $k \times 1$ vectors of piecewise continuous functions.

The vector functions $\psi_{m1}(\zeta_1, z)$ and $\psi_{m2}(z, \zeta_2)$ defined by (3.4.44) or (3.4.45) depending on whether ζ is unbounded or bounded respectively are integrals of vectors of absolute values.* To simplify the analysis of this section we write these vector functions in the form

$$(C.13) \quad \begin{aligned} \psi_{m1}(\zeta_1, z) &= \int_{\zeta_1}^z C_{m1}(t) |dt| \\ \psi_{m2}(z, \zeta_2) &= \int_z^{\zeta_2} C_{m2}(t) |dt| \end{aligned}$$

Using the definitions (3.4.43) for α, β, γ and δ together with the results of the previous section, we obtain from (3.4.34)

$$(C.14) \quad \begin{bmatrix} |W_{jm1}^v(z)| \\ |W_{jm2}^v(z)| \end{bmatrix} \leq \begin{bmatrix} \int_{\zeta_1}^z \{ [\alpha |W_{jm1}^v(t)| + \beta |W_{jm2}^v(t)|] \left| \frac{(1-\eta)}{t^{2-\eta}} \right| + C_{m1}(t) \} |dt| \\ \int_z^{\zeta_2} \{ [\gamma |W_{jm1}^v(t)| + \delta |W_{jm2}^v(t)|] \left| \frac{(1-\eta)}{t^{2-\eta}} \right| + C_{m2}(t) \} |dt| \end{bmatrix}$$

By Section 3.4.2, the contour $\mathcal{P}$ with end points ζ_1 and ζ_2 is mapped onto the real interval $[a, b]$, the points $t = \zeta_1, z$, and ζ_2 corresponding to the points $s = a, \sigma$ and b respectively. Hence with $\omega(s)ds = \left| \frac{(1-\eta)dt}{t^{2-\eta}} \right|$, $\theta_k(s)ds = C_{mk}(t) |dt|$, $|W_{jmk}^v(z)| = \varphi_k(\sigma)$, ($k = 1, 2$), the equations (C.14) may be written

$$(C.15) \quad \begin{aligned} \varphi_1(\sigma) &\leq \int_a^\sigma \{ [\alpha \varphi_1(s) + \beta \varphi_2(s)] \omega(s) + \theta_1(s) \} ds \\ \varphi_2(\sigma) &\leq \int_\sigma^b \{ [\gamma \varphi_1(s) + \delta \varphi_2(s)] \omega(s) + \theta_2(s) \} ds \end{aligned}$$

* Variations of vectors over a contour.

On applying Lemma C.1 to each equation in (C.15) we obtain

$$(C.16) \quad \begin{aligned} \varphi_1(\sigma) &\leq \int_a^\sigma \exp \left\{ \int_s^\sigma \alpha \omega(\tau) d\tau \right\} \{ \beta \varphi_2(s) \omega(s) + \theta_1(s) \} ds \\ \varphi_2(\sigma) &\leq \int_\sigma^b \exp \left\{ \int_\sigma^s \delta \omega(\tau) d\tau \right\} \{ \gamma \varphi_1(s) \omega(s) + \theta_2(s) \} ds \end{aligned}$$

Substituting the bound for $\varphi_2(s)$ in the second of (C.16) for $\varphi_2(s)$ in the first we obtain

$$(C.17) \quad \begin{aligned} \varphi_1(\sigma) &\leq \int_a^\sigma \exp \left\{ \alpha \int_s^\sigma \omega(\tau) d\tau \right\} \left[\beta \int_s^b \exp \left\{ \delta \int_s^u \omega(\tau) d\tau \right\} \{ \gamma \varphi_1(u) \omega(u) + \right. \\ &\quad \left. + \theta_2(u) \} du \omega(s) + \theta_1(s) \right] ds \end{aligned}$$

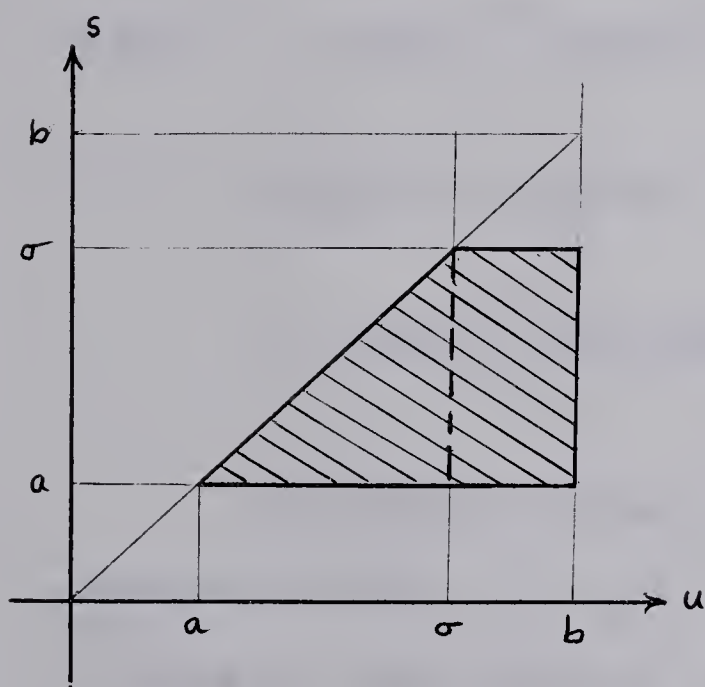


FIGURE 4

The repeated integral (C.17) may be interpreted as a double integral over the shaded region indicated in Figure 4. Splitting up the integral as an integral over a triangle plus an integral over a rectangle and interchanging the order of integration in each case we obtain

$$(C.18) \quad \begin{aligned} \varphi_1(\sigma) &\leq \theta_3(\sigma) \\ &+ \int_a^\sigma \int_a^\tau \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \beta \exp \left\{ \delta \int_s^\tau \omega(v) dv \right\} \{ \gamma \varphi_1(\tau) \omega(\tau) + \theta_2(\tau) \} \omega(s) ds d\tau \\ &+ \int_\sigma^b \int_a^\sigma \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \beta \exp \left\{ \delta \int_s^\tau \omega(v) dv \right\} \{ \gamma \varphi_1(\tau) \omega(\tau) + \theta_2(\tau) \} \omega(s) ds d\tau \end{aligned}$$

where we have put

$$(C.19) \quad \theta_3(\sigma) = \int_a^\sigma \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \theta_1(s) ds$$

On evaluating the inner integrals in (C.18) we find that

$$(C.20) \quad \phi_1(\sigma) \leq \theta_3(\sigma) + \left\{ \int_a^\sigma X_1(\sigma, \tau) + \int_\sigma^b X_2(\sigma, \tau) \right\} \{ \gamma \phi_1(\tau) \omega(\tau) + \theta_2(\tau) \} d\tau$$

where $X_1(\sigma, \tau)$ and $X_2(\sigma, \tau)$ are given by the solutions of

$$(C.21) \quad \begin{aligned} & \alpha X_1(\sigma, \tau) + X_1(\sigma, \tau) \delta \\ &= \exp \left\{ \alpha \int_a^\sigma \omega(s) ds \right\} \left[\beta \exp \left\{ \delta \int_a^\tau \omega(s) ds \right\} - \exp \left\{ -\alpha \int_a^\tau \omega(s) ds \right\} \beta \right] \\ & \alpha X_2(\sigma, \tau) + X_2(\sigma, \tau) \delta \\ &= \left[\exp \left\{ \alpha \int_a^\sigma \omega(s) ds \right\} \beta - \beta \exp \left\{ -\delta \int_a^\sigma \omega(s) ds \right\} \right] \exp \left\{ \delta \int_a^\tau \omega(s) ds \right\} \end{aligned}$$

According to Bellman [11] both of the above equations have unique solutions provided $\mu_i + \nu_j \neq 0$ ($i = 1, 2, \dots, \kappa$; $j = 1, 2, \dots, n-\kappa$), μ_i and ν_j being the eigen values of α and δ respectively, and where α is a $\kappa \times \kappa$ matrix. By another theorem of Bellman [2], if this condition is not satisfied, it can always be made satisfied by an arbitrarily small change in the elements of the matrices α and δ . Nevertheless an exact explicit expression of the right hand side of (C.20) is not practical for obtaining error bounds, and we make a slight sacrifice of sharpness of error bound in order to achieve a matrix bound which is much simpler to evaluate. Several

simplifications of the bounds are possible at this point; one of these is as follows:

$$\begin{aligned}
 (C.22) \quad & \int_a^\tau \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \beta \exp \left\{ \delta \int_s^\tau \omega(v) dv \right\} \omega(s) ds \\
 & \leq \int_a^\tau \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \beta \exp \left\{ \delta \int_a^\tau \omega(v) dv \right\} \omega(s) ds \\
 & = \alpha^{-1} \left[\exp \left\{ \alpha \int_a^\sigma \omega(u) du \right\} - \exp \left\{ \alpha \int_\tau^\sigma \omega(u) du \right\} \right] \beta \exp \left\{ \delta \int_a^\tau \omega(v) dv \right\}^* \\
 & \leq \alpha^{-1} \left[\exp \left\{ \alpha \int_a^\sigma \omega(u) du \right\} - I_\kappa \right] \beta \exp \left\{ \delta \int_a^\tau \omega(v) dv \right\}
 \end{aligned}$$

Similarly

$$\begin{aligned}
 (C.23) \quad & \int_a^\sigma \exp \left\{ \alpha \int_s^\sigma \omega(u) du \right\} \beta \exp \left\{ \delta \int_s^\tau \omega(v) dv \right\} \omega(s) ds \\
 & \leq \alpha^{-1} \left[\exp \left\{ \alpha \int_a^\sigma \omega(u) du \right\} - I_\kappa \right] \beta \exp \left\{ \delta \int_a^\tau \omega(v) dv \right\} \\
 & \equiv G(\sigma) H(\tau)
 \end{aligned}$$

where

$$\begin{aligned}
 (C.24) \quad & G(\sigma) = \alpha^{-1} \left[\exp \left\{ \alpha \int_a^\sigma \omega(u) du \right\} - I_\kappa \right] \\
 & H(\tau) = \exp \left\{ \delta \int_a^\tau \omega(v) dv \right\}
 \end{aligned}$$

* The possible difficulty arising as a result of the vanishing of the determinant of α is avoided if we remember that the exponentials are defined by power series expansions.

Thus combining the above, we have

$$(C.25) \quad \varphi(\sigma) \leq \theta_3(\sigma) + \int_a^b G(\sigma) H(\tau) \{ \gamma \varphi(\tau) \omega(\tau) + \theta_2(\tau) \} d\tau$$

On applying Lemma C.3 to equation (C.25) and transform back to complex variable notation we obtain the results of Theorem 3.4.4 (and (3.3.4)).

Let us now proceed to prove Theorem 3.4.5 (and 3.3.5). For this purpose we let norms α , β , γ and δ be defined as in Theorem 3.4.3. If we define non-negative scalar continuous functions φ_1 , φ_2 , ω , θ_1 and θ_2 , and θ_3 by $\varphi_k(\sigma) = \|W_{jmk}^\nu(z)\|$, $\varphi_1(s)ds = |d\psi_1(\xi_1, t)|$, $\varphi_2(s)ds = |d\psi_2(t, \xi_2)|$ (equation (3.4.42)) $\omega(s)ds = \left| \frac{(1-\eta)dt}{t^{2-\eta}} \right| = \left| \frac{(1-\eta)}{t(s)^{2-\eta}} \frac{dt(s)}{ds} \right| ds$; and θ_3 is defined using the definitions of this paragraph together with (C.19). Again $a \leq s \leq b$.

Starting with equations (C.15) where, instead of having vector inequalities we now have scalar inequalities, the analysis proceeds exactly as above to equation (C.21). Since however, the elements α , β , γ and δ along with $X_1(\sigma, \tau)$ and $X_2(\sigma, \tau)$ are now scalar quantities, we are able to explicitly solve the equations (C.21), to obtain

$$(C.26) \quad \varphi(\sigma) \leq \theta_3(\sigma) + \int_a^b X(\tau, \sigma) \{ \gamma \varphi(\tau) \omega(\tau) + \theta_2(\tau) \} d\tau$$

where

$$(C.27) \quad X(\tau, \sigma) = \frac{\beta}{\alpha + \delta} \left[\exp \left\{ \alpha \int_a^\sigma \omega(s) ds + \delta \int_a^\tau \omega(s) ds \right\} - \exp \left\{ \alpha \int_\tau^\sigma \omega(s) ds \right\} \right], a \leq \tau \leq \sigma$$

$$\frac{\beta}{\alpha + \delta} \left[\exp \left\{ \alpha \int_a^\sigma \omega(s) ds + \delta \int_a^\tau \omega(s) ds \right\} - \exp \left\{ \delta \int_\sigma^\tau \omega(s) ds \right\} \right], 0 \leq \tau \leq b.$$

We over-estimate the bound on the elements $X(\tau, \sigma)$ in order to obtain an integral equation which satisfies Lemma C.3. Over the whole range of (a, b) each of the functions in (C.27) satisfies

$$(C.28) \quad X(\tau, \sigma) \leq X^*(\sigma) = \frac{2\beta}{\alpha + \delta} \exp \left\{ \Omega \int_a^b \omega(s) ds + \right. \\ \left. + \left| \frac{\alpha - \delta}{2} \right| \int_a^\sigma \omega(s) ds \right\} \sinh \left\{ \frac{\alpha + \delta}{2} \int_a^\sigma \omega(s) ds \right\} .$$

where $\Omega = \max(\alpha, \delta)$.

On substituting the right of (C.28) into (C.26) we again obtain an integral inequality of the form (C.25); applying Lemma C.3 to this inequality, the results of Theorem 3.4.5 follow.

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